1. Let V be the set of vectors in R^3 given by $\{(a,b,c) \text{ in } R^3 \mid a=-c \text{ and } b=0\}$.

(a). Does V contain the 0 vector? Yes.

$(0,0,0)$ satisfies that $a=-c$ $(0=-0)$, and $b=0$

(b). Is V closed under addition? Yes.

Say $u: (a_1, b_1, c_1)$ and $v: (a_2, b_2, c_2)$ are in V.

then, we must have $b_1=b_2=0$, $a_1=-c_1$, $a_2=-c_2$

Then, as $a_1+a_2=-c_1-c_2$, $b_1+b_2=0$, so $u+v: (a_1+a_2, b_1+b_2, c_1+c_2)$ is still in V.

(c). Is V closed under multiplication? Yes.

Say $u: (a_1, b_1, c_1)$ is in V.

then, we must have $a_1=-c_1$, $b_1=0$.

Then, if we multiply a real number k to u: $ku=(ka_1, kb_1, kc_1)$,

where $k \cdot b_1=0$, and $ka_1=-kc_1$.

So, ku is still in V.

(d). Is V a subspace of R^3? Yes.

Based on definition: ① Zero-vector is in V — meet

② Closure under addition — meet

③ Closure under scaler multiplication — meet

So, V is a subspace of R^3.

2. Let V be the set of vectors in R^3 given by $\{(a,b,c) \text{ in } R^3 \mid c \geq 0\}$

(a). Does V contain the 0 vector? Yes.

(0,0,0) satisfies that $c \geq 0$

(b). Is V closed under addition? Yes.

Say $u: (a_1, b_1, c_1)$ and $v: (a_2, b_2, c_2)$ are in V.

then we must have $c_1 \geq 0$, $c_2 \geq 0$

Then, as $c_1 + c_2 \geq 0$, so $u+v: (a_1+a_2, b_1+b_2, c_1+c_2)$ is still in V.

(c). Is V closed under multiplication? No

Say $u: (a_1, b_1, c_1)$ is in V.

then, we must have $c_1 \geq 0$.

Then, if we multiply a real number k to u: $ku = (ka_1, kb_1, kc_1)$,

where kc_1 might be negative (i.e. $k=-1$), in which case ku is not in V.

(d). Is V a subspace of R^3? No

Based on definition: ① Zero-vector is in V — meet

② Closure under addition — meet

③ Closure under scaler multiplication — does not meet.

So, V is not a subspace of R^3.

3. When is the set of solutions to a matrix equation a subspace?

Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

In this case $x=0$, y and z are free variables.

The set of solutions are: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z$

So, span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ meets the requirement of ① $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in the subspace

② closed under addition

③ closed under scaler multiplication.

So, span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a subspace in R^3

Counter-Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

In this case $x=1$, y and z are free variables.

The set of solutions are: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

The set of solutions is not a subspace because $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is not in the set.

Combining both examples, we have the choice "Sometimes".

4. Observe that in the row-reduced matrix:

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

"Red" representing pivots.

So, first, second, third columns are pivotal columns.

Since, the pivot columns of a matrix A form a basis for Col (A)

As we go back to the previous matrix, the first, second, third columns are the vectors spanning Col(A).

So, the solution should be: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$

Note: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ does not span the Col(A), because Col(A) contains

vectors vectors whose last coordinate is nonzero.

5. A's row reduced echelon form is as follows: $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

The free variables are x_2 and x_4. The parametric form of the solution set is:

$$\begin{cases} x_1 = -x_4 \\ x_2 = x_2 \\ x_3 = -2x_4 \\ x_4 = x_4 \end{cases} \xrightarrow[\text{vector form}]{\text{parametric}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} x_4$$

Therefore, $\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$