Quiz Solutions 10/2/20

Which of the following sets ~~ar~~ is a basis for a plane in $\mathbb{R}^3$? Circle all that apply.

- $\left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \\ 3 \end{pmatrix} \right\}$ is not a basis for a plane in $\mathbb{R}^3$ since $-3\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \\ 3 \end{pmatrix}$, i.e. the set is linearly dependent. A basis for a plane requires two linearly independent vectors.

- $\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \right\}}$ is a basis for a plane in $\mathbb{R}^3$, $\left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 2 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$, so $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix}$ are linearly independent. Since a basis for a plane requires two linearly independent vectors, this set is a basis for a plane in $\mathbb{R}^3$.

- $\left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right\}$ is not a basis for a plane in $\mathbb{R}^3$.
$\left(\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$, so $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ are all linearly independent vectors, and form a basis for a plane in $\mathbb{R}^3$.

Consider the 3x5 matrix

$$A = \begin{pmatrix} 3 & -6 & -8 & -8 & 5 \\ -6 & 12 & 3 & 3 & 3 \\ -4 & 8 & 0 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The <u>rank</u> of A is $\boxed{2}$ Since rank = dim(Col(A)), where Col(A) is the span of the linearly independent vectors in matrix A.

The <u>nullity</u> of A is $\boxed{3}$ Since nullity = dim(Nul(A)), where Nul(A) is the span of the linearly independent vectors after parameterizing Matrix A.

For a 3x5 matrix, the rank plus the nullity is always equal to $\boxed{5}$

By the <u>Rank Theorem</u>: dim(Col(A)) + dim(Nul(A)) = rank + nullity, which equals the number of columns of matrix $A_{m \times n}$.

Answer the following True/False questions.

(a) It is possible to have a 4x6 matrix whose rank is 3 and whose nullity is 1.

$\boxed{\text{FALSE}}$ By the Rank Theorem for $A_{m\times n}$, rank + nullity $= n$.

Rank $= \dim(\mathrm{Col}(A)) = 3$ nullity $= \dim(\mathrm{Nul}(A)) = 1$

$3+1 = 4 \neq 6$

(b) If a set of four vectors span $\mathbb{R}^4$, then the set is a basis for $\mathbb{R}^4$.

$\boxed{\text{TRUE}}$ If a set of vectors span $\mathbb{R}^4$, the set is linearly independent. Any vector in $\mathbb{R}^4$ ~~and~~ may then be formed a linear combination of these vectors. Hence, this set is a basis for $\mathbb{R}^4$.

Consider the matrix $A = \begin{pmatrix} 1 & -1 \\ 6 & 1 \\ 1 & 6 \end{pmatrix}$. Let T be the matrix transformation $T(\vec{v}) = A\vec{v}$.
Find a vector $\vec{x}$ where $T(\vec{x})$ is equal to $\begin{pmatrix} 1 \\ -1 \\ -6 \end{pmatrix}$.

The answer is $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ where $\boxed{a = 0}$ $\boxed{b = -1}$

We only want the second column of A so the first entry in $\vec{x}$ must be 0 by matrix multiplication. The second entry in $\vec{x}$ must be -1 since we want $(-1)\begin{pmatrix} -1 \\ 1 \\ 6 \end{pmatrix}$.

That is, $\begin{pmatrix} 1 & -1 \\ 6 & 1 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -6 \end{pmatrix}$.

Consider the matrix $A = \begin{pmatrix} 3 & -6 & -8 & -8 & 5 \\ -6 & 12 & 3 & 3 & 3 \\ -4 & 8 & 0 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.
Let T be the matrix transformation $T(\vec{v}) = A(\vec{v})$.

The domain of T is $\mathbb{R}^n$, where $n = \boxed{5}$ (the domain corresponds to the number of columns in $A_{m\times n}$).

The codomain of T is $\mathbb{R}^m$, where $m = \boxed{3}$ (the codomain corresponds to the number of rows in $A_{m\times n}$).

The range of T has dimension $\boxed{2}$ (the dimension of the range equals the dimension of the column space).