Quiz 7 Solutions

- Suppose that a, b, c, and d are real numbers and that

① $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$

Compute the determinant of

② $\begin{pmatrix} 5a-7c & 5b-7d \\ a & b \end{pmatrix}$

Solution

From det for matrix ①, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 = ad - cb$

From matrix ② . . .

$$\det\begin{pmatrix} 5a-7c & 5b-7d \\ a & b \end{pmatrix} = b(5a-7c) - a(5b-7d)$$
$$= \cancel{5ab} - 7cb - \cancel{5ab} + 7ad$$
$$= +7ad - 7cb$$
$$= 7(\underbrace{ad - cb}_{=1})$$
$$= 7(1) = 7$$

- Find value of h that makes the matrix not invertible:

$A = \begin{pmatrix} 0 & 6 & 6 \\ -1 & 7 & 2 \\ -2 & 6 & h \end{pmatrix}$

Solution

*matrix is not invertible if $\det(A) = 0$
*we can solve for the $\det(A)$ by cofactor expansion.
*we can use this pattern for co-factor expansion:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det\begin{pmatrix} 0 & 6 & 6 \\ -1 & 7 & 2 \\ -2 & 6 & h \end{pmatrix} = +0\begin{vmatrix} 7 & 2 \\ 6 & h \end{vmatrix} - 6\begin{vmatrix} -1 & 2 \\ -2 & h \end{vmatrix} + 6\begin{vmatrix} -1 & 7 \\ -2 & 6 \end{vmatrix}$$
$$= 0 - 6(-h+4) + 6(-6+14)$$

(cont. to next page)

$= 6h - 24 - 36 + 84 = 0$
$= 6h = -24$
$h = -4$

• T or F

a) For any two 2×2 matrices A and B we have
$\det(A+B) = \det(A) + \det(B)$

Solution => False !!
* Keyword => any
* $\det(A+B) \neq \det(A) + \det(B)$

Example to prove why this statement is false:

$A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 6 & 2 \\ 7 & 8 \end{pmatrix}$

$\det(A) = 15 - 2 = 13 \qquad \det(B) = 48 - 14 = 34$

$A + B = \begin{pmatrix} 9 & 3 \\ 9 & 13 \end{pmatrix} \qquad \det(A+B) = 9(13) - 27 = 117$

$\det(A+B) \neq \det(A) + \det(B)$
$117 \neq 13 + 34$
$117 \neq 47$

b) For any 2×2 matrix A we have $\det(-A) = \det(A)$

Solution => True !!

Example to prove:

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \det(A) = 3$

$-A = (-1)\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}$

$\det(-A) = 4 - 1 = 3$

$\det(A) = \det(-A)$ ✓

if $A = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \quad \det(A) = ** - **$

$-A = \begin{pmatrix} -* & -* \\ -* & -* \end{pmatrix} \quad \det(-A) = (-*)(-*) - (-*)(-*) = (**) - (**)$

*the negative signs will always cancel out for the 2×2 matrix to make $\det(-A) = \det(A)$

• Find a value of $\underline{h}$ that makes the statement $\underline{\text{true}}$

$$\det\begin{pmatrix} h-5 & -7 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix} = 48$$

Solution

* Solve using co-factor expansion

$$\det\begin{pmatrix} h-5 & -7 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix} = 48 = (h-5)\begin{vmatrix} 4 & 2 \\ 0 & 3 \end{vmatrix} - (-7)\begin{vmatrix} 0 & 2 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix}$$

$$= (h-5)(12-0) + 7(\cancel{0}-0) + \cancel{0}$$

$$= 12h - \overset{\text{"60"}}{\cancel{50}} = 48$$

$$12h = 108$$

$$h = 9$$

or

$$\det\begin{pmatrix} h-5 & -7 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{pmatrix} = 48 = 3\begin{vmatrix} h-5 & -7 \\ 0 & 4 \end{vmatrix}$$

$$48 = 3((h-5)4 - 0)$$

$$48 = 3(4h-20)$$

$$48 = 12h - 60$$

$$108 = 12h$$

$$h = 9$$

• Let S be a square in $\mathbb{R}^2$ whose corners are $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. For each matrix below, consider how the following matrix transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

For which matrices does $T(S)$ have area 2? (*Select all that apply*)

<u>Solution</u>

* The absolute value of the determinant of a 2×2 matrix = area

☒ $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \rightarrow \det\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 4-1 = 3 \neq 2$

☑ $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \det\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -1-1 = |-2| = 2$ ✓

☒ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \rightarrow \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4-0 = 4 \neq 2$

☒ $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \det\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 0-0 = 0 \neq 2$

☑ $\begin{pmatrix} 3 & 7 \\ 1 & 3 \end{pmatrix} \rightarrow \det\begin{pmatrix} 3 & 7 \\ 1 & 3 \end{pmatrix} = 9-7 = 2 = 2$ ✓

☒ $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \det\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} = \sqrt{2} - 0 = \sqrt{2} \neq 2$