Reduced Row Echelon Form

fails rule 4

Poll

Which are in reduced row echelon form?

REF not RREF

NO $\left(\begin{array}{c|c} \boxed{1} & 0 \\ 0 & \boxed{2} \end{array}\right)$ not 1

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Yes

L not there

Not even REF

No $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ fails 1

$\begin{pmatrix} 0 & \boxed{1} & 0 & 0 \end{pmatrix}$ yes

$\begin{pmatrix} 0 & \boxed{1} & 8 & 0 \end{pmatrix}$ yes

$\left(\begin{array}{cc|c} \boxed{1} & 17 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right)$ yes

$\begin{pmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ yes

REF:

1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

RREF:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

Announcements Aug 26

- WeBWorK on Section 1.1 due Thursday night
- Quiz on Section 1.1 Friday 8 am - 8 pm EDT
- My office hours Tue 11-12, Thu ~~2-3~~, and by appointment

  1-2 ← new weekly time.
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Juntao Thu 3-4
- Studio on Friday
- Stay tuned for PLUS session info
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Teams/Piazza
- Find a group to work with - let me know if you need help

Section 1.2

Row reduction

Row Reduction and Echelon Forms

A matrix is in row echelon form if

1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is easy to solve using back substitution.

The pivot positions are the leading entries in each row.

Reduced Row Echelon Form

A system is in reduced row echelon form if also:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

For example:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is even easier to solve.

Important. In any discussion of row echelon form, we ignore any vertical lines!

Can every matrix be put in reduced row echelon form?

Row Reduction Algorithm

To find row echelon form:

Step 1 Swap rows so a leftmost nonzero entry is in 1st row (if needed)

Step 2 Scale 1st row so that its leading entry is equal to 1

Step 3 Use row replacement so all entries below this 1 (or, pivot) are 0

Then cover the first row and repeat the three steps.

To then find reduced row echelon form:

- Use row replacement so that all entries above the pivots are 0.

Examples.

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \left(\begin{array}{rrr|r} 0 & 7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right) \left(\begin{array}{rrr|r} 4 & -5 & 3 & 2 \\ 1 & -1 & -2 & -6 \\ 4 & -4 & -14 & 18 \end{array}\right)$$

Interactive Row Reducer

Solutions of Linear Systems

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

What are the solutions? Say the variables are x and y.

$x = 5$

$y = 2$

$(5, 2)$

Solutions of Linear Systems: Consistency

Solve the linear system associated to:

$$\begin{array}{r c} & \begin{array}{cccc} x & y & z & \end{array} \\ \begin{array}{r} 1^{st} \text{ eqn} \\ 2^{nd} \text{ eqn} \end{array} & \left(\begin{array}{ccc|c} \boxed{1} & 0 & 5 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right) \end{array}$$

$x + 5z = 0$

$0 = 1$ X

Inconsistent!

= no soln.

Say the variables are x, y, and z.

A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

gives: $0 = 1$

Example with a parameter

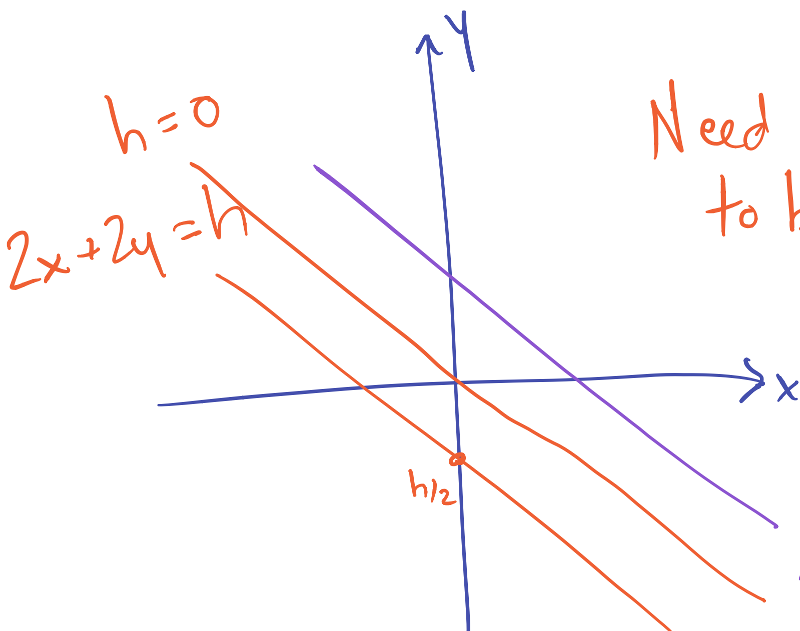
For which values of h does the following system have a solution?

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= h \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & h \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} \boxed{1} & 1 & 1 \\ 0 & 0 & \boxed{h-2} \end{array}\right)$$

Want no pivot in last col.
So $h-2=0$
So $h=2$

Solve this by row reduction and also solve it by thinking geometrically.



Need the lines to be same

$\rightsquigarrow h=2.$

1.3 Parametric Form

Outline of Section 1.3

- Find the parametric form for the solutions to a system of linear equations.
- Describe the geometric picture of the set of solutions.

Free Variables

We know how to understand the solution to a system of linear equations when every column to the left of the vertical line has a pivot. For instance:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

2 mins ago.

$(5, 2)$

If the variables are x and y what are the solutions?

Free Variables

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? For instance:

$$\begin{array}{r c} & \begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array} \\ \begin{array}{r} \text{1st eqn} \\ \text{2nd eqn} \end{array} & \left(\begin{array}{ccc|c} \boxed{1} & 0 & 5 & 0 \\ 0 & \boxed{1} & 2 & 1 \end{array}\right) \end{array}$$

no pivot.

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 5 & 0 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right)$$

no pivot

represents two equations:

$$\boxed{\begin{aligned} x_1 \quad\quad\; + 5x_3 &= 0 \\ x_2 + 2x_3 &= 1 \end{aligned}}$$

$\cancel{x_3 = 1}$

There is one free variable x_3, corresponding to the non-pivot column.
To solve, we move the free variable to the right:

all answers:

$$\begin{aligned} x_1 \quad\quad &= -5x_3 \\ x_2 &= 1 - 2x_3 \\ x_3 &= x_3 \text{ (free; any real number)} \end{aligned}$$

Find one answer:

$x_3 = 7$

$x_2 = -13$

$x_1 = -35$

$(-35, -13, 7)$

This is the parametric solution. We can also write the solution as:

free vars = dim of soln.

$$(-5x_3, 1 - 2x_3, x_3)$$

line

What is one particular solution? What does the set of solutions look like?

Free Variables

Solve the system of linear equations in x_1, x_2, x_3, x_4:

$$\begin{array}{rcrcrcrcl} x_1 & & & + & 5x_3 & & & = & 0 \\ & & & & & & x_4 & = & 0 \end{array}$$

So the associated matrix is:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

To solve, we move the free variable to the right:

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= \quad x_2 \qquad \text{(free)} \\ x_3 &= \quad x_3 \qquad \text{(free)} \\ x_4 &= 0 \end{aligned}$$

2D -

Or: $\boxed{(-5x_3, x_2, x_3, 0)}$. This is a plane in $\mathbb{R}^4$.

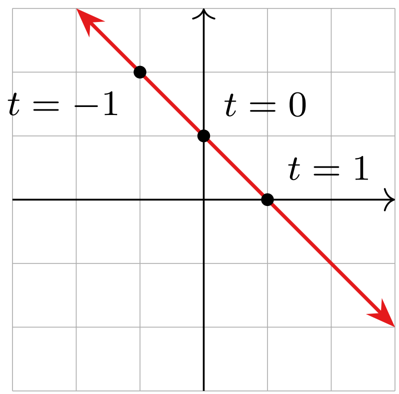
The original equations are the implicit equations for the solution. The answer to this question is the parametric solution.

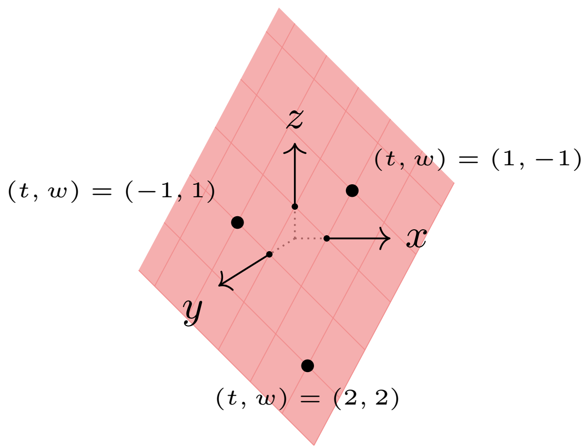
Free variables

Geometry

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional plane in $\mathbb{R}^n$.

Why does this make sense?





Poll

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. two points
4. line
5. plane
6. 3-dimensional plane
7. 4-dimensional plane

Implicit versus parametric equations of planes

Find a parametric description of the plane

$$x + y + z = 1$$

Step 1

Make a matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right)$$

Step 2

RREF

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \end{array}\right)$$

free free.

Step 3

Find free vars, etc.

$x = 1 - y - z$

$y = y$ (free)

$z = z$ (free)

$$(1 - y - z, y, z)$$

The original version is the implicit equation for the plane. The answer to this problem is the parametric description.

Also correct:

$$(x, 1 - x - z, z)$$

But doesn't follow our recipe.

Typical exam questions

True/False: If a system of equations has 100 variables and 70 equations, then there must be infinitely many solutions.

False.

Maybe inconsistent.

True/False: If a system of equations has 70 variables and 100 equations, then it must be inconsistent.

False

Maybe all eqns same (or multiples)

How can we tell if an augmented matrix corresponds to a consistent system of linear equations?

No pivot in last column.

If a system of linear equations has finitely many solutions, what are the possible numbers of solutions?

0 or 1

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of system of linear equations.

1. The last column is a pivot column.
   $\rightsquigarrow$ the system is *inconsistent*.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   $\rightsquigarrow$ the system has a *unique solution*.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   $\rightsquigarrow$ the system has *infinitely many* solutions; free variables correspond to columns without pivots.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

RREF

4x4 aug mat, not a pivot in each col, inconsistent

Want pivots in cols 1, 2, 4

need to be 0.

$0=1$

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 4 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

incons.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

cons.