**Poll**

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. two points
4. line
5. plane
6. 3-dimensional plane
7. 4-dimensional plane

# Announcements Aug 31

- WeBWorK on Sections 1.2 and 1.3 due Thursday night
- Quiz on Sections 1.2 and 1.3 Friday 8 am - 8 pm EDT
- My office hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Juntao Thu 3-4
- Studio on Friday
- Stay tuned for PLUS session info
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

# Chapter 2

## System of Linear Equations: Geometry

# Where are we?

In Chapter 1 we learned how to completely solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution.

In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning.

# Section 2.1

## Vectors

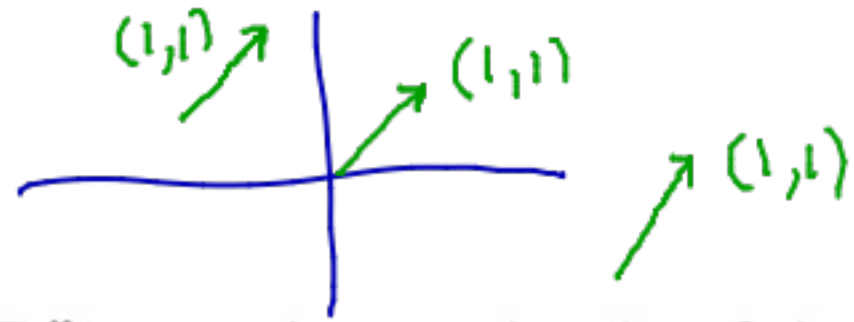
# Outline

- Think of points in $\mathbb{R}^n$ as vectors.
- Learn how to add vectors and multiply them by a scalar
- Understand the geometry of adding vectors and multiplying them by a scalar
- Understand linear combinations algebraically and geometrically

# Vectors

A vector is a matrix with one row or one column. We can think of a vector with $n$ rows as:

- a point in $\mathbb{R}^n$
- an arrow in $\mathbb{R}^n$



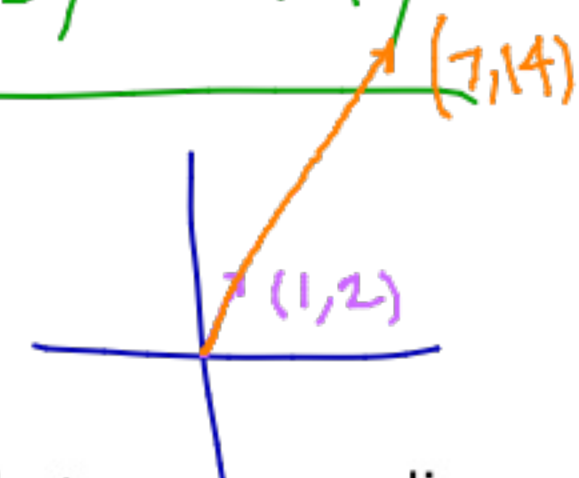

To go from an arrow to a point in $\mathbb{R}^n$, we subtract the tip of the arrow from the starting point. Note that there are many arrows representing the same vector.

Adding vectors / parallelogram rule Demo

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

Scaling vectors Demo

$$7 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$



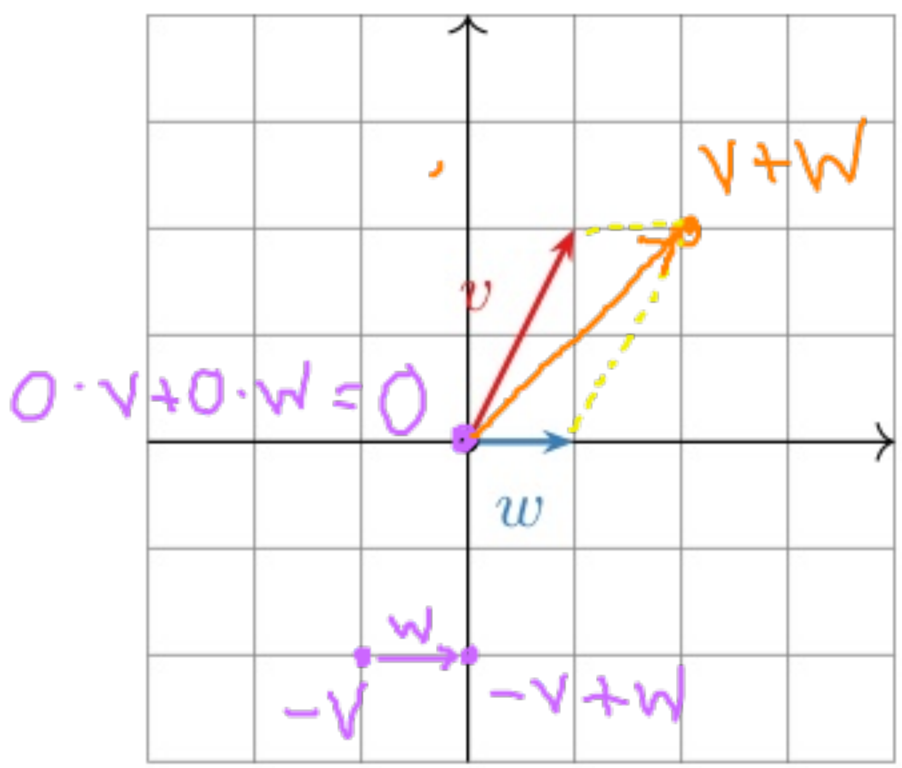

A scalar is just a real number. We use this term to indicate that we are scaling a vector by this number.

# Linear Combinations

A linear combination of the vectors $v_1, \ldots, v_k$ is any vector

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

where $c_1, \ldots, c_k$ are real numbers.




$\cdot\ 2v+3w$

identify $\begin{pmatrix} a \\ b \end{pmatrix}$ with $(a,b)$ in $\mathbb{R}^2$

Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of $v$ and $w$?

$v + w = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$c_1 = 1 \quad c_2 = 1$

$1 \cdot v + 1 \cdot w$

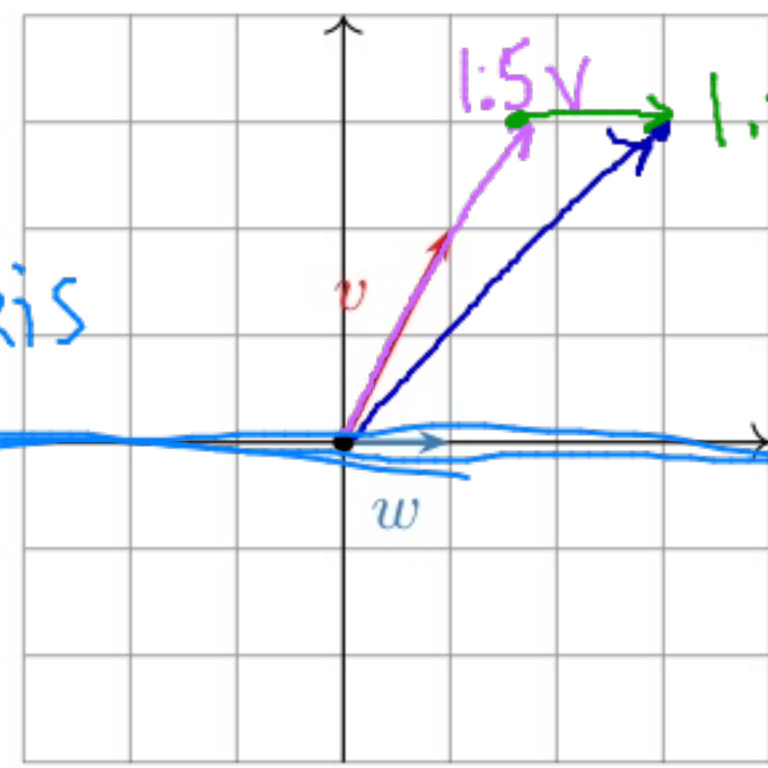
**Poll**

Is there a vector in $\mathbb{R}^2$ that is not a linear combination of $v$ and $w$?

- yes
- no

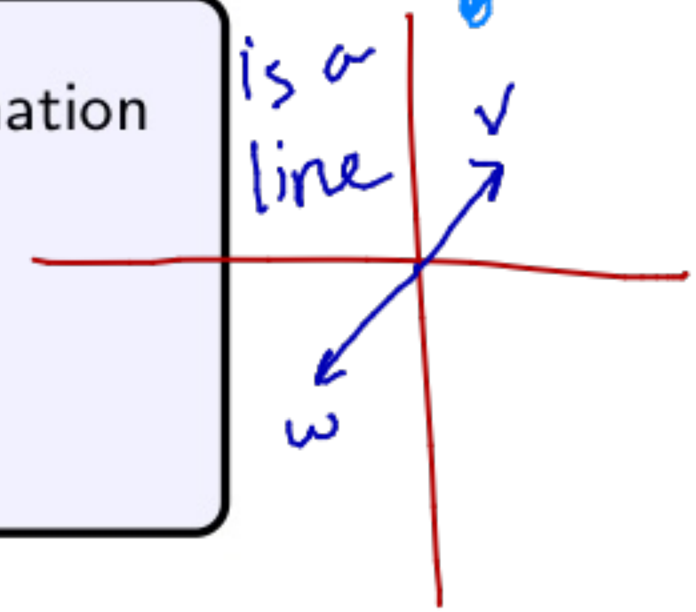
Span

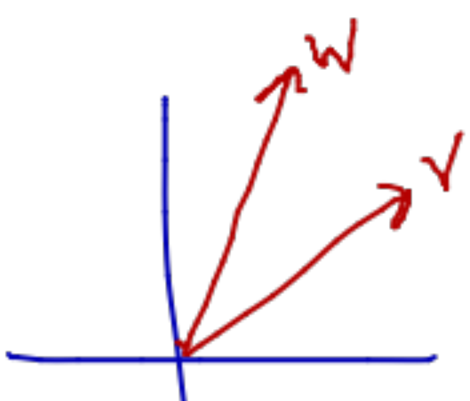
Set of lin. combos of w = x-axis




$$1.5\begin{pmatrix}1\\2\end{pmatrix} + 1.5\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$$

Set of lin. combos is a line




Set of lin combos is $\mathbb{R}^2$

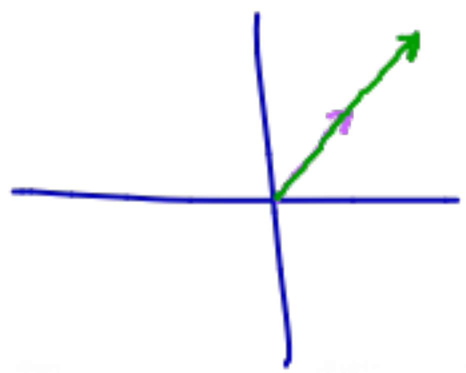
Set of linear combos of v & w is $\mathbb{R}^2$

# Linear Combinations

What are some linear combinations of $(1, 1)$?

(a,a) line y=X

What are some linear combinations of $(1, 1)$ and $(2, 2)$?



Same answer

What are some linear combinations of $(0, 0)$?

origin

(0,0)

# Summary of Section 2.1

- A vector is a point/arrow in $\mathbb{R}^n$
- We can add/scale vectors algebraically & geometrically (parallelogram rule)
- A linear combination of vectors $v_1, \ldots, v_k$ is a vector

$$c_1 v_1 + \cdots + c_k v_k$$

  where $c_1, \ldots, c_k$ are real numbers.

# Typical exam questions

True/False: For any collection of vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, the zero vector in $\mathbb{R}^n$ is a linear combination of $v_1, \ldots, v_k$.

True/False: The vector $(1, 1)$ can be written as a linear combination of $(2, 2)$ and $(-2, -2)$ in infinitely many ways.

Suppose that $v$ is a vector in $\mathbb{R}^n$, and consider the set of all linear combinations of $v$. What geometric shape is this?

# Section 2.2

## Vector Equations and Spans

# Outline of Section 2.2

- Learn the equivalences:

  $$\text{vector equations} \leftrightarrow \text{augmented matrices} \leftrightarrow \text{linear systems}$$

- Learn the definition of span
- Learn the relationship between spans and consistency

## Linear Combinations

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

vector eqn.

$$x\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

sys of lin eqns

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned}$$

Write down an equation in order to solve this problem. This is called a vector equation.

row reduce!

Notice that the vector equation can be rewritten as a system of linear equations. Solve it!

matrix eqn

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} \boxed{1} & -1 & 8 \\ 0 & \boxed{5} & -45 \\ 0 & 0 & 0 \end{array}\right)$$

# Linear combinations, vector equations, and linear systems

In general, asking:

Is $b$ a linear combination of $v_1, \ldots, v_k$?

is the same as asking if the vector equation

$$x_1 v_1 + \cdots + x_k v_k = b$$

is consistent, which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).

# Span

Essential vocabulary word!

$$\begin{aligned}\text{Span}\{v_1, v_2, \ldots, v_k\} &= \{x_1v_1 + x_2v_2 + \cdots + x_kv_k \mid x_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

What are the possibilities for the span of two vectors in $\mathbb{R}^2$?

Demo

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo

Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

# Span

Essential vocabulary word!

$$\begin{aligned}\mathrm{Span}\{v_1, v_2, \ldots, v_k\} &= \{x_1v_1 + x_2v_2 + \cdots x_kv_k \mid x_i \text{ in } \mathbb{R}\} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

Four ways of saying the same thing:

- $b$ is in Span$\{v_1, v_2, \ldots, v_k\}$ $\leftarrow$ geometry
- $b$ is a linear combination of $v_1, \ldots, v_k$
- the vector equation $x_1v_1 + \cdots + x_kv_k = b$ has a solution $\leftarrow$ algebra
- the system of linear equations corresponding to

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Demo Demo

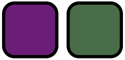
## Span

Is $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$?

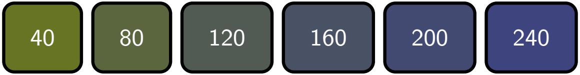
# Application: Additive Color Theory

Consider now the two colors

$$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$$



For which $h$ is $(116, 130, h)$ in the span of those two colors?



# Summary of Section 2.2

- vector equations $\leftrightarrow$ augmented matrices $\leftrightarrow$ linear systems
- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.
- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.

# Typical exam questions

Is $\begin{pmatrix} 8 \\ 16 \\ 1 \end{pmatrix}$ in the span of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

Write down the vector equation for the previous problem.

True/False: The vector equation $x_1 v_1 + \cdots + x_k v_k = 0$ is always consistent.

True/False: It is possible for the span of 3 vectors in $\mathbb{R}^3$ to be a line.

True/False: the plane $z = 1$ in $\mathbb{R}^3$ is a span.