Poll

$D = \begin{pmatrix} 0 & 5 \\ 0 & 1 \end{pmatrix}$ → diag'able: 2 distinct eigenvals

Not invertible.

Poll

Which are diagonalizable?

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

no

alg mult of 1 is 2

geom mult of 1 is 1

yes 2 distinct eigenvals

yes. it is already diagonal.

$E = I E I^{-1}$

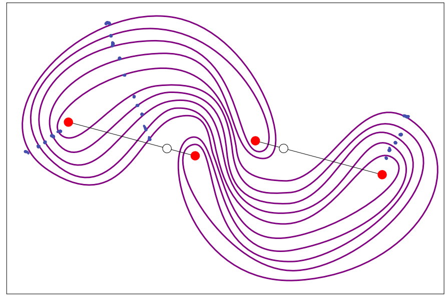
↑ diagonal

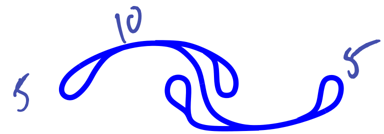
Announcements Nov 4

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on 5.1, 5.2 due Thursday night
- Quiz on Sections 5.1, 5.2 Friday 8 am - 8 pm EDT
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- Writing assignment due Nov 24 (make sure to read the emails I sent)
- My Office Hours Tue 11-12, Thu 9-10, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`

Taffy pullers

How efficient is this taffy puller?





If you run the taffy puller, the taffy starts to look like the shape on the right. Every rotation of the machine changes the number of strands of taffy by a matrix:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

The largest eigenvalue λ of this matrix describes the efficiency of the taffy puller. With every rotation, the number of strands multiplies by λ.

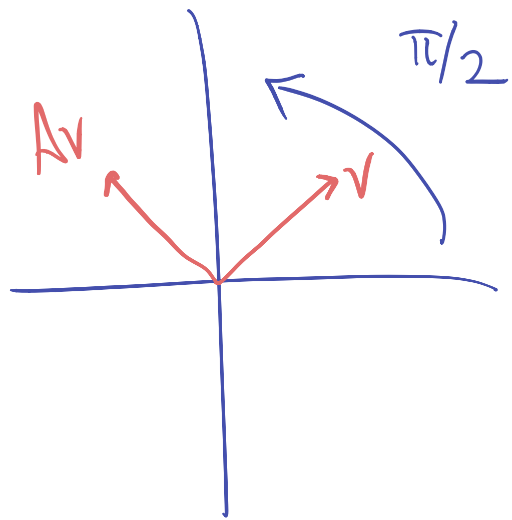
Section 5.5

Complex Eigenvalues

Outline of Section 5.5

- Rotation matrices have no eigenvectors
- Crash course in complex numbers
- Finding complex eigenvectors and eigenvalues
- Complex eigenvalues correspond to rotations + dilations

Demo Demo



A matrix without an eigenvector

Recall that rotation matrices like

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

rotation + Scale by $\sqrt{2}$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

have no eigenvectors. Why? ✓

Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can't take the square root of a negative number:

$$x^2 + 1 = 0$$

$x^2 = -1$

$x = \pm\sqrt{-1}$

$x = \pm i$

Solution. Take square roots of negative numbers:

$$x = \pm\sqrt{-1}$$

We usually write $\sqrt{-1}$ as i (for "imaginary"), so $x = \pm i$.

Now try solving these:

$$x^2 + 3 = 0$$

$x^2 = -3$

$x = \pm\sqrt{-3} = \pm\sqrt{3}\sqrt{-1}$

$= \pm\sqrt{3}\, i$

$$x^2 - x + 1 = 0$$

$x = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm \sqrt{3}\, i}{2}$

Complex numbers

We can add/multiply (and divide!) complex numbers:

$(2 - 3i) + (-1 + i) =$ $1 - 2i$

$(2 - 3i)(-1 + i) =$ $2(-1) + 2i - 3i(-1) - 3i^2$

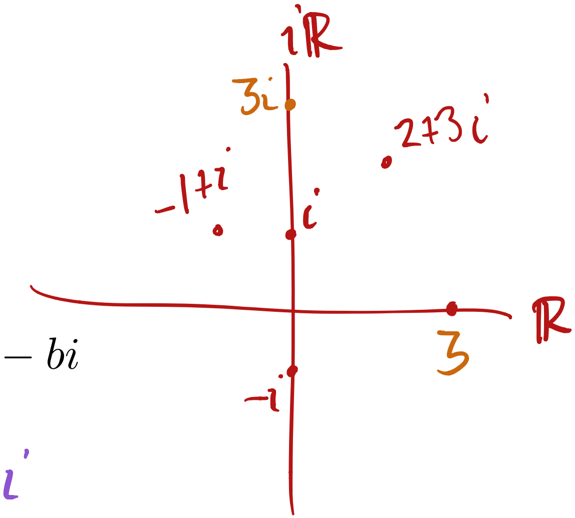
$-2 + 2i + 3i + 3$

$1 + 5i$

-1

$$\frac{2-3i}{-1+i} \cdot \frac{-1-i}{-1-i}$$

Complex numbers



The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can conjugate complex numbers: $\overline{a + bi} = a - bi$

$$\overline{5+7i} = 5-7i$$

$$\overline{5-7i} = 5+7i$$

$$\overline{1+i} = 1-i$$

$$\overline{-1-i} = -1+i$$

$$\overline{5} = 5$$

$$\overline{-5} = -5$$

Complex numbers and polynomials $(\lambda - \text{root})(\lambda - \text{root})(\lambda - \text{root})$

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots (counted with multiplicity).

Fact. If z is a root of a real polynomial then $\overline{z}$ is also a root.

So what are the possibilities for degree 2, 3 polynomials? $\longrightarrow$ deg 2

2 real roots

2 complex roots

What does this have to do with eigenvalues of matrices?

deg 3

1 or 3 real roots

2 or 0 complex

Pf of Fact: $p(x)$ real polynomial

$$p(z) = 0$$

$$p(\overline{z}) = \overline{p(z)} = \overline{0} = 0$$

Complex eigenvalues

Say A is a square matrix with real entries.

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

$v = \begin{pmatrix} 1+i \\ 5+7i \end{pmatrix}$ $\overline{v} = \begin{pmatrix} 1-i \\ 5-7i \end{pmatrix}$

Why?

If $Av = \lambda v$

then $\overline{Av} = \overline{\lambda v}$

$\overline{A}\,\overline{v} = \overline{\lambda}\,\overline{v}$

$A\overline{v} = \overline{\lambda}\,\overline{v}$

$\overline{ab} = \overline{a}\,\overline{b}$

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

rotation by $\pi/2$

Char poly: $\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 \rightsquigarrow \pm i$

$\underline{\lambda = i}$ $\quad \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix}$

trick #1

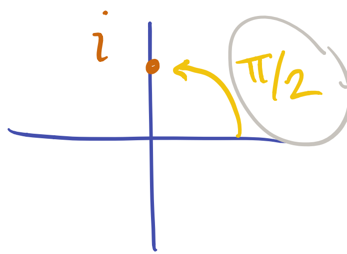
eigenvector: $\begin{pmatrix} -1 \\ +i \end{pmatrix}$

trick #2

$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} b \\ -a \end{pmatrix}$

$\underline{\lambda = -i}$

eigenvector: $\begin{pmatrix} -1 \\ -i \end{pmatrix}$



Three shortcuts for complex eigenvectors

Suppose we have a 2×2 matrix with complex eigenvalue λ.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

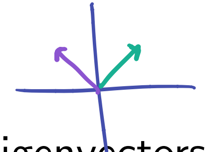
$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

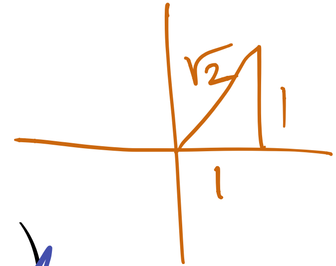
the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.

Complex eigenvalues

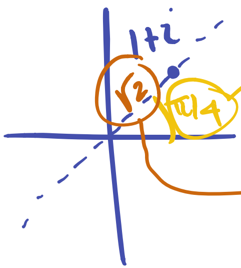




Find the complex eigenvalues and eigenvectors for

guess eigenvalue

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$



π/4 rot
Scale by √2

eigenvals: $\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} \rightarrow (\lambda-1)^2+1 = \lambda^2-2\lambda+2$

$$\frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

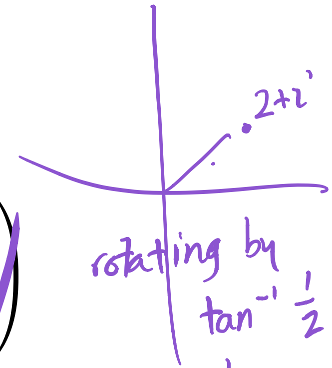
$\underline{\lambda = 1+i}$ $\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow{\text{trick\#1}} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix}$

trick #2 $\rightarrow \begin{pmatrix} -1 \\ +i \end{pmatrix}$

$\underline{\lambda = 1-i}$ $\begin{pmatrix} -1 \\ -i \end{pmatrix}$

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for



$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

rotating by $\tan^{-1}\frac{1}{2}$

scaling by $\sqrt{5}$

eigenvals $\begin{pmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix}$

$(3-\lambda)(1-\lambda)+2$

$\lambda^2-4\lambda+5$

$\dfrac{4\pm\sqrt{-4}}{2} = 2\pm i$

$\lambda = 2-i$ $\begin{pmatrix} 1-(2-i) & -2 \\ 1 & 3-(2-i) \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1+i & -2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} +2 \\ -1+i \end{pmatrix}$

$\lambda = 2+i$ $\begin{pmatrix} 2 \\ -1-i \end{pmatrix}$

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\cancel{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} \quad \cancel{\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

rotation by $\pi/2$ & scale by 2

By looking:

$\lambda = 1 \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = \pm 2i$.

Recipe

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & -2 \\ 0 & 2 & -\lambda \end{pmatrix} = (1-\lambda)(\lambda^2+4)$$

$\lambda = 1 \quad \lambda = \pm 2i$

$$\underline{\lambda = 1} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 2 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 2i \quad \begin{pmatrix} 1-2i & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 2 & -2i \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ -2 \\ +2i \end{pmatrix}$$

Summary of Section 5.5

- Complex numbers allow us to solve all polynomials completely, and find n eigenvectors for an $n \times n$ matrix
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$

v is a λ-eigenvector

means $v \neq 0$ & v in $\mathrm{Nul}(A - \lambda I)$

$$A - \lambda I \xrightarrow[\text{red}]{\text{row}} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & y/x \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} -y/x \\ 1 \end{pmatrix} \quad \begin{pmatrix} -y \\ x \end{pmatrix}$$

Typical Exam Questions 5.5

- True/False. If v is an eigenvector for A with complex entries then $i \cdot v$ is also an eigenvector for A.
- True/False. If $(i, 1)$ is an eigenvector for A then $(i, -1)$ is also an eigenvector for A.
- If A is a 4×4 matrix with real entries, what are the possibilities for the number of non-real eigenvalues of A?
- Find the eigenvalues and eigenvectors for the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$