Announcements Nov 9

- Today's class is happening early. Watch the recording at 3:30 if you can't make it.
- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on 5.4, 5.5 due Thursday night
- Quiz on Sections 5.4, 5.5 Friday 8 am - 8 pm EDT
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- Writing assignment due Nov 24 (make sure to read the emails I sent)
- Office hours this week Tuesday 4-5, Thursday 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`

Section 5.6

Stochastic Matrices (and Google!)

Outline of Section 5.6

- Stochastic matrices and applications
- The steady state of a stochastic matrix
- Important web pages

Stochastic matrices

A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.

Examples:

$$\begin{pmatrix} 1/4 & 3/5 \\ 3/4 & 2/5 \end{pmatrix} \quad \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} \quad \begin{pmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 0 & 1/4 \\ 0 & 0 & 1/4 \end{pmatrix}$$

Application: Rental Cars

Say your car rental company has 3 locations. Make a matrix whose ij entry is the fraction of cars at location j that end up at location i. For example,

start at location 1

end at loc 1 →
end at 2 →
end at 3 →

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix} = A$$

Note the columns sum to 1. Why? ✓

$$\begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix}$$

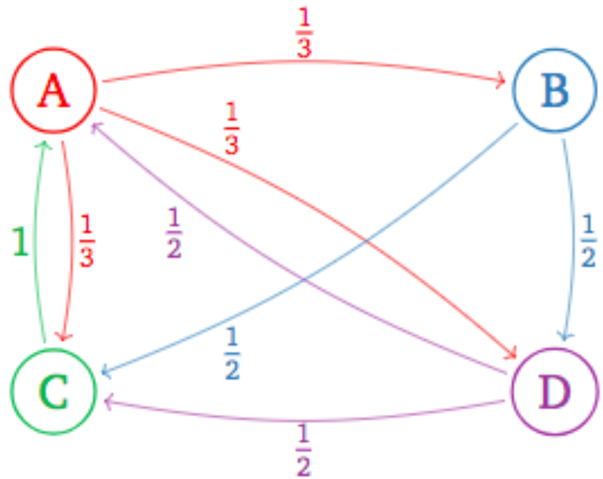
Say I start with 100 cars at each location & every car gets rented each day.

After 1 day: $A\begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 120 \\ 100 \\ 80 \end{pmatrix}$

After 2 days: $A\begin{pmatrix} 120 \\ 100 \\ 80 \end{pmatrix} = \begin{pmatrix} \; \end{pmatrix}$

Application: Web pages

Make a matrix whose ij entry is the fraction of (randomly surfing) web surfers at page j that end up at page i. If page i has N links then the ij-entry is either 0 or $1/N$.



$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Properties of stochastic matrices

A has eigenval 1 means: is v so $Av = v$.

Let A be a stochastic matrix.

Fact. One of the eigenvalues of A is 1 and all other eigenvalues have absolute value at most 1.

Why?

$$A = \begin{pmatrix} 1/4 & 2/5 \\ 3/4 & 3/5 \end{pmatrix}$$

Can see: A^T has eigenval 1

$$\begin{pmatrix} 1/4 & 3/4 \\ 2/5 & 3/5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

A & A^T have same eigenvals (same char poly) but diff. eigenvecs.

Properties of stochastic matrices

Let A be a positive stochastic matrix, meaning all entries are positive. (no zeros).

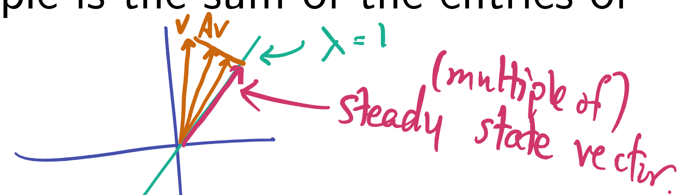
Same Fact. One of the eigenvalues of A is 1 and all other eigenvalues have absolute value at most 1 (same as before).

Fact. The 1-eigenspace of A is 1-dimensional; it has a positive eigenvector. example: $\begin{pmatrix}5\\3\\1\end{pmatrix}$

The unique such eigenvector with entries adding to 1 is called the steady state vector. above example: $\begin{pmatrix}5/9\\3/9\\1/9\end{pmatrix}$ $5+3+1=9$.

Fact. Under iteration, all nonzero vectors approach a multiple of the steady state vector. The multiple is the sum of the entries of the original vector.

Demo



The last fact tells us how to distribute rental cars, and also tells us the importance of web pages!

Example

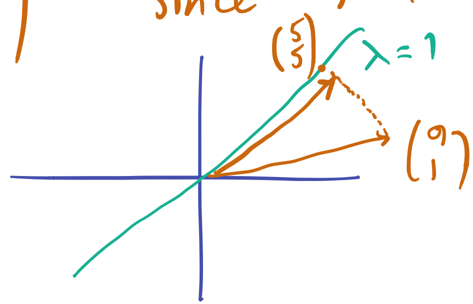
Find the steady state vector.

$$A = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}$$

To what vector does $A^n \left(\begin{smallmatrix} 1 \\ 9 \end{smallmatrix}\right)$ approach as $n \to \infty$

$\lambda = 1$ $\begin{pmatrix} -3/4 & 3/4 \\ 3/4 & -3/4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$(1+9) \cdot \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ since $9 + 1 = 10$.



Application: Rental Cars

The rental car matrix is:

3/10

$$\begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Its steady state vector is:

$$\require{cancel}\cancel{\frac{1}{18}}\begin{pmatrix} 7/18 \\ 6/18 \\ 5/18 \end{pmatrix} \approx \begin{pmatrix} .39 \\ .33 \\ .28 \end{pmatrix}$$

If we start with this distribution, the distribution never changes.

$$A^n \begin{pmatrix} 100 \\ 100 \\ 100 \end{pmatrix} \longrightarrow 300 \cdot \begin{pmatrix} .39 \\ .33 \\ .28 \end{pmatrix}$$

So more cars end up at location 1.

Application: Web pages

The web page matrix is:

$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Its steady state vector is approximately

$$\begin{pmatrix} .39 \\ .13 \\ .29 \\ .19 \end{pmatrix}$$

and so the first web page is the most important.

Fine print

There are a couple of problems with the web page matrix as given:

- What happens if there is a web page with no links?
- What if the internet graph is not connected?
- How do you find eigenvectors for a huge matrix?

Here are the solutions:

- Make a column with $1/n$ in each entry
  (the surfer goes to a new page randomly).
- Let B be the matrix with all entries equal to 1, replace A with

$$.85 * A + .15 * B$$

- Approximate via iteration! v = any vector

v, Av, A^2v, A^3v approximates.

Summary of Section 5.6

- A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.
- Every stochastic matrix has 1 as an eigenvalue, and all other eigenvalues have absolute value at most 1.
- A positive stochastic matrix has 1-dimensional eigenspace and has a positive eigenvector.
- For a positive stochastic matrix, a positive 1-eigenvector with entries adding to 1 is called a steady state vector.
- For a positive stochastic matrix, all nonzero vectors approach the steady state vector under iteration.
- Steady state vectors tell us the importance of web pages (for example).

Typical Exam Questions 5.6

- Is there a stochastic matrix where the 1-eigenspace has dimension greater than 1?
- Find the steady state vector for this matrix:

$$A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 1/3 \end{pmatrix}$$

  To what vector does $A^n \left(\begin{smallmatrix} 5 \\ 7 \end{smallmatrix}\right)$ approach as $n \to \infty$?
- Find the steady state vector for this matrix:

$$A = \begin{pmatrix} 1/3 & 1/5 & 1/4 \\ 1/3 & 2/5 & 1/2 \\ 1/3 & 2/5 & 1/4 \end{pmatrix}$$

- Make your own internet and see if you can guess which web page is the most important. Check your answer using the method described in this section.