Announcements Nov 11

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on 5.4, 5.5 due Thursday night
- Quiz on Sections 5.4, 5.5 Friday 8 am - 8 pm EDT
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- Writing assignment due Nov 24 (make sure to read the emails I sent)
- Office hours this week Tuesday 4-5, Thursday 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`

Note: Part of 5.5 got cut
Dynamics
Block diag.

Chapter 6

Orthogonality

Section 6.1

Dot products and Orthogonality

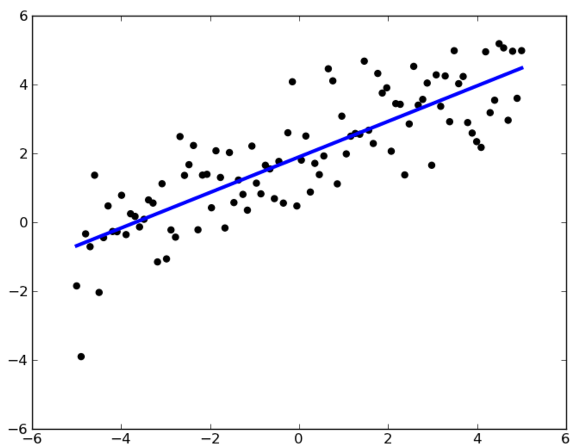
Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

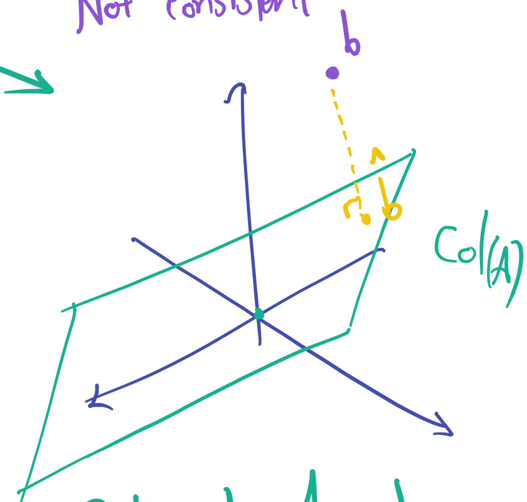
We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?

$\hat{b}$ is closest pt to b in Col(A)



Not consistent



Can't solve $Ax = b$.
Solve $Ax = \hat{b}$ instead.

The answer relies on orthogonality.

Outline

- Dot products
- Length and distance
- Orthogonality

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$\begin{aligned} u \cdot v &= \sum_{i=1}^{n} u_i v_i \\ &= u_1 v_1 + \cdots + u_n v_n \\ &= u^T v \end{aligned}$$

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$. $= \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

$= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$

Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \Leftrightarrow u = 0$

$u \cdot u = u_1^2 + u_2^2 + \cdots + u_n^2$

Length

Let v be a vector in $\mathbb{R}^n$

$$\|v\| = \sqrt{v \cdot v}$$
$$= \text{length of } v$$

$\|v\|^2 = v \cdot v.$

Why? Pythagorean Theorem

$$\left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\right\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$

Fact. $\|cv\| = |c| \cdot \|v\|$

scalar mult

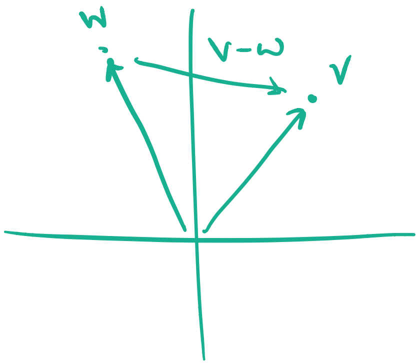
v is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $\underbrace{(1,2,3,4)}_{=v}$.

$$\text{answer} = \frac{v}{\|v\|} = \frac{1}{\sqrt{1^2+2^2+3^2+4^2}}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \frac{1}{\sqrt{30}}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}1/\sqrt{30}\\2/\sqrt{30}\\3/\sqrt{30}\\4/\sqrt{30}\end{pmatrix}$$

Distance

The distance between v and w is the length of $v - w$ (or $w - v$!).



Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$.

$$\left\| \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\4\\-3\end{pmatrix} \right\| = \left\| \begin{pmatrix}0\\-3\\4\end{pmatrix} \right\| = \sqrt{0^2+3^2+4^2}$$

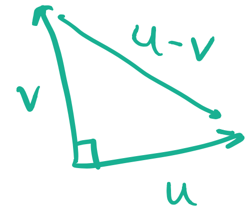
$$= \sqrt{25} = 5$$

Orthogonality

Fact. $u \perp v \Leftrightarrow u \cdot v = 0$

$\perp$ means orthogonal/perpendicular

Why? Pythagorean theorem again!



$$
\begin{aligned}
u \perp v &\Leftrightarrow \|u\|^2 + \|v\|^2 = \|u - v\|^2 \\
&\Leftrightarrow \cancel{u \cdot u} + \cancel{v \cdot v} = \cancel{u \cdot u} - 2u \cdot v + \cancel{v \cdot v} \\
&\Leftrightarrow u \cdot v = 0
\end{aligned}
$$

Problem. Find a nonzero vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.

$$\begin{pmatrix} -13 \\ 2 \\ 3 \end{pmatrix} \perp \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{since} \quad \begin{pmatrix} -13 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$$

$$-13 + 4 + 9 = 0.$$

Summary of Section 6.1

- $u \cdot v = \sum u_i v_i$
- $u \cdot u = \|u\|^2$ (length of u squared)
- The unit vector in the direction of v is $v/\|v\|$.
- The distance from u to v is $\|u - v\|$
- $u \cdot v = 0 \Leftrightarrow u \perp v$

Section 6.2

Orthogonal complements

Outline of Section 6.2

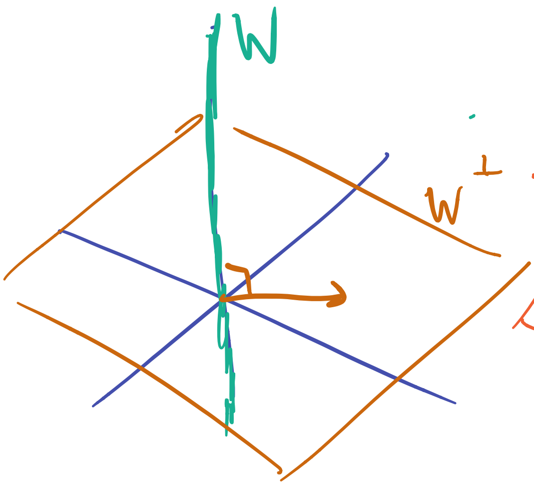
- Orthogonal complements
- Computing orthogonal complements

Orthogonal complements

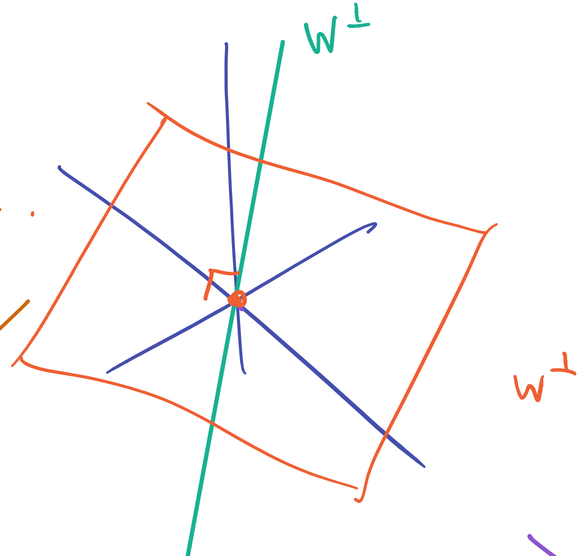
W = subspace of $\mathbb{R}^n$ = plane thru 0.

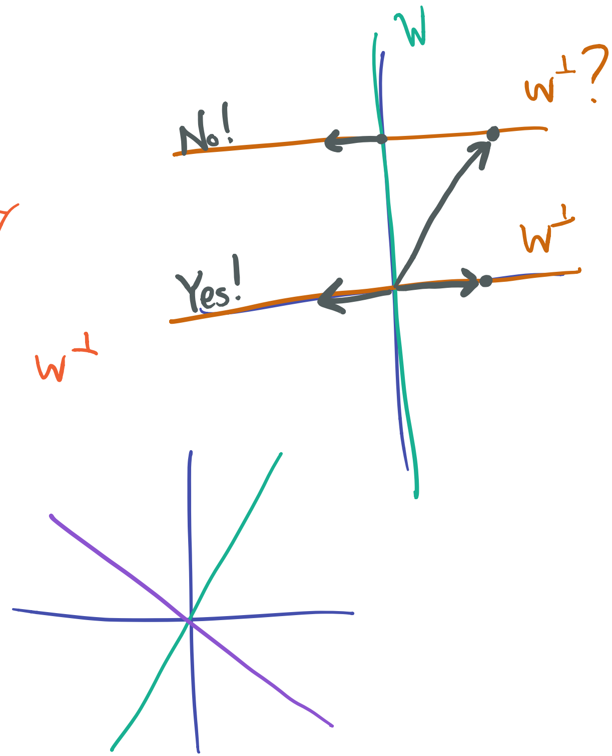
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?



Demo Demo





Orthogonal complements

W = subspace of $\mathbb{R}^n$

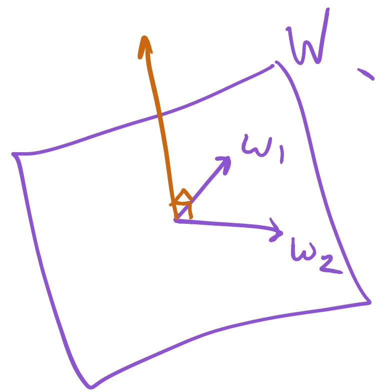
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.

$u \cdot w = 0 \rightsquigarrow cu \cdot w = 0$

$u \cdot w = 0$ & $v \cdot w = 0 \rightsquigarrow (u+v) \cdot w = 0$.



Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1)\}$. Find the equation of the plane $W^\perp$.

What equation(s) are we solving?

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $W^\perp$ means: $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$.

$$\boxed{x + y - z = 0.}$$

$\mathrm{Nul}\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$

Find a basis for $W^\perp$.

$x = -y + z$
$y = y$
$z = z$

$y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find a system of equations describing the line $W^\perp$.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $W^\perp$ means: $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$$\boxed{\begin{aligned} x + y - z &= 0 \\ -x + 2y + z &= 0 \end{aligned}}$$

$$\boxed{\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0}$$

Find a basis for $W^\perp$.

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \leadsto \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$x = z$

$y = 0$

$z = z$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

$$\boxed{\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0}$$

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find a system of equations describing the line $W^\perp$.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

row space is W

Theorem. $A = m \times n$ matrix

$$(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A$$

$$\boxed{\begin{pmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0}$$

Geometry $\leftrightarrow$ Algebra

(The row space of A is the span of the rows of A.)

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.

Why? Say that $w_1 + w_1' = w_2 + w_2'$ where w_1 and w_2 are in W and w_1' and w_2' are in $W^\perp$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in W and the latter is in $W^\perp$, so they must both be equal to 0.

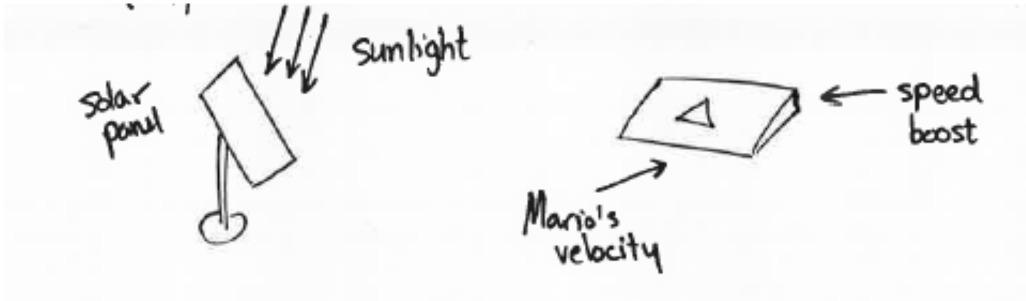
Demo

Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Summary of Section 6.2

- $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$
- Facts:
  1. $W^\perp$ is a subspace of $\mathbb{R}^n$
  2. $(W^\perp)^\perp = W$
  3. $\dim W + \dim W^\perp = n$
  4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
     $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
  5. The intersection of W and $W^\perp$ is $\{0\}$.
- To find $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find the null space.
- Every vector v can be written uniquely as $v = v_W + v_{W^\perp}$ with v_W in W and $v_{W^\perp}$ in $W^\perp$

Typical Exam Questions 6.2

- What is the dimension of $W^{\perp}$ if W is a line in $\mathbb{R}^{10}$?
- What is $W^{\perp}$ if W is the line $y = mx$ in $\mathbb{R}^2$?
- If W is the x-axis in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write v as $v_W + v_{W^{\perp}}$.
- If W is the line $y = x$ in $\mathbb{R}^2$, and $v = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, write v as $v_W + v_{W^{\perp}}$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^3$.
- Find a basis for the orthogonal complement of the line through $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ in $\mathbb{R}^4$.
- What is the orthogonal complement of x_1x_2-plane in $\mathbb{R}^4$?