Announcements Nov 18

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on 5.6, 6.1 due Thursday night
- WeBWorKs 6.2, 6.2, and 6.5 not graded
- Writing assignment due Nov 24 (make sure to read the emails I sent)
- No more quizzes
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- Final Exam Dec 4 Friday Dec 4 9 am - 9 pm (cumulative)
- Review Sessions tba Midterm 3 Review Thu 3:30 with Manuel
- Office hours Tuesday 11-12, Thursday 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- No more studio
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`

Poll on Piazza now!

Chapter 4

Determinants

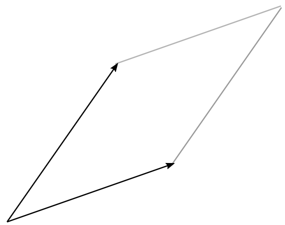
Section 4.1

The definition of the determinant

Invertibility and volume

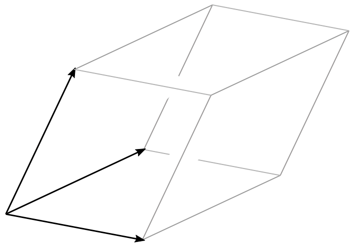
When is a 2×2 matrix invertible? $\leftarrow$ Algebra

When the rows (or columns) don't lie on a line $\Leftrightarrow$ the corresponding parallelogram has non-zero area. $\leftarrow$ Geometry



When is a 3×3 matrix invertible?

When the rows (or columns) don't lie on a plane $\Leftrightarrow$ the corresponding parallelepiped (3D parallelogram) has non-zero volume



Same for $n \times n$!

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.

Properties of the determinant

Fact 1. There is such a number $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$ ok, now we need to say what transpose is

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

If you want the proofs, see the book. Actually Fact 1 is the hardest!

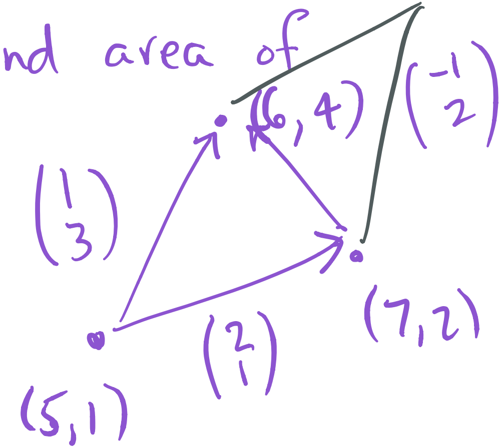
Powers

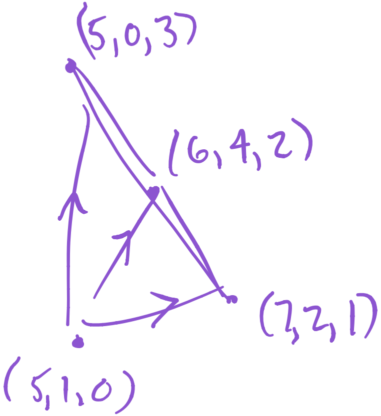
Poll

Suppose we know A^5 is invertible. Is A invertible?

1. yes
2. no
3. maybe

Find area of





$$\frac{\left|\det\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}\right|}{2} = \frac{|-5|}{2} = \frac{5}{2}$$

Section 4.2

Cofactor expansions

A formula for the determinant

We will give a recursive formula.

First some terminology:

$A_{ij} = ij$th minor of A
$= (n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column

$C_{ij} = (-1)^{i+j} \det(A_{ij})$
$= ij$th cofactor of A

Finally:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Or:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

So we find the determinant of a 3×3 matrix in terms of the determinants of 2×2 matrices, etc.

Determinants

Consider

$$A = \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Compute the following:

$a_{11} =$ $a_{12} =$ $a_{13} =$

$A_{11} =$ $A_{12} =$ $A_{13} =$

$\det A_{11} =$ $\det A_{12} =$ $\det A_{13} =$

$C_{11} =$ $C_{12} =$ $C_{13} =$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Determinants

Poll

What is the determinant?

$$\det \begin{pmatrix} 4 & 7 & 0 & 9 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 5 & 9 & 2 & 10 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

A formula for the inverse

(from Section 3.3)

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$n \times n$ matrices

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \dots & C_{nn} \end{pmatrix} \\ &= \frac{1}{\det(A)} (C_{ij})^T \end{aligned}$$

Check that these agree!

The proof uses Cramer's rule (see the notes on the course home page. We're not testing on this - it's just for your information.)

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.

Typical Exam Questions 4.2

- True or false. The cofactor expansion across the first row gives the negative of the cofactor expansion across the second row.
- Find the determinant of the following matrix using one of the formulas from this section:

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & 0 & 9 \end{pmatrix}$$

- Find the determinant of the following matrix using one of the formulas from this section:

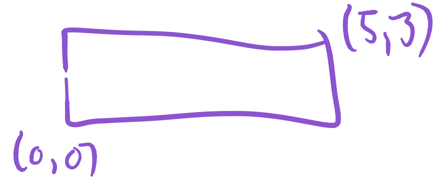
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$

- Find the cofactor matrix for the above matrix and use it to find the inverse.

Section 4.3

The determinant and volumes

Determinants and linear transformations



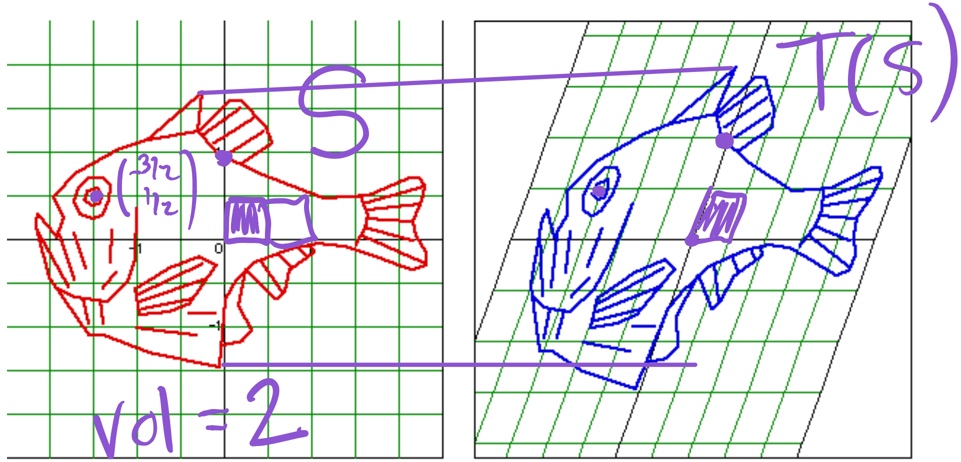
Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

$\begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 1/2 \end{pmatrix}$

This works even if S is curvy, like a circle or an ellipse, or:



T is a shear. $\begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}$

Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

$$\mathrm{Vol}\, T(S) = \left|\det\begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}\right| \cdot \mathrm{Vol}(S) = 1 \cdot 2 = 2$$

Summary of Sections 4.1 and 4.3

Say $\det$ is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S)$.

Typical Exam Questions 4.1 and 4.3

- Find the value of h that makes the determinant 0:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & h \end{pmatrix}$$

- If the matrix on the left has determinant 5, what is the determinant of the matrix on the right?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{pmatrix}$$

- If the area of a fish (in a photo) is 7 square inches, and we apply a shear, what is the new area?
- Suppose that T is a linear transformation with the property that $T \circ T = T$. What is the determinant of the standard matrix for T?
- Suppose that T is a linear transformation with the property that $T \circ T = \text{identity}$. What is the determinant of the standard matrix for T?
- Find the volume of the triangular pyramid with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 0)$, and $(1, 2, 3)$.

Chapter 5

Eigenvectors and eigenvalues

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or}$$
> $$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), fluid mixing, RLC circuits, clustering (data analysis), principal component analysis, Google, Netflix (collaborative prediction), infectious disease models, special relativity, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

A Question from Biology

In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year - think of it as a vector (f, s, t) - what is the population the next year?

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix}$$

Now choose some starting population vector $u = (f, s, t)$ and choose some number of years N. What is the new population after N years?

Demo

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

In simpler terms: Av is a scalar multiple of v.

In other words: Av points in the same direction as v.

Think of this in terms of inputs and outputs!

eigen = characteristic (or: self)

This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

When we apply large powers of the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

to a vector v not on the x-axis, we see that $A^n v$ gets closer and closer to the y-axis, and it's length gets approximately tripled each time. This is because the largest eigenvalue is 3 and its eigenspace is the y-axis.

For the rabbit matrix

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

We will see that 2 is the largest eigenvalue, and its eigenspace is the span of the vector $(16, 4, 1)$. That's why all populations of rabbits tend towards the ratio 16:4:1 and why the population approximately doubles each year.

Eigenvectors and Eigenvalues

Confirming eigenvectors

Poll

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?

Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

- Reflection about a line in $\mathbb{R}^2$ (doesn't matter which line!)
- Orthogonal projection onto a line in $\mathbb{R}^2$ (doesn't matter which line!)
- Scaling of $\mathbb{R}^2$ by 3
- (Standard) shear of $\mathbb{R}^2$
- Orthogonal projection to a plane in $\mathbb{R}^3$ (doesn't matter which plane!)

Demo

Summary of Section 5.1

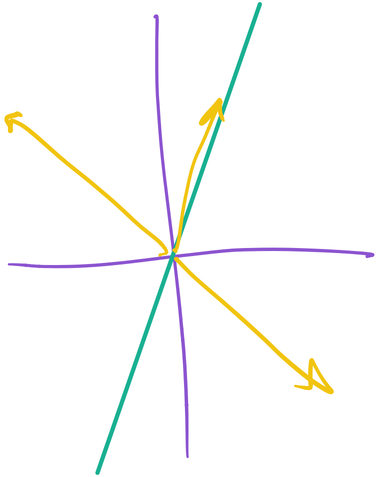
- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- Recipe: The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A
- Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations

Typical exam questions 5.1

- Find the 2-eigenvectors for the matrix

$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

- True or false: The zero vector is an eigenvector for every matrix.
- How many different eigenvalues can there be for an $n \times n$ matrix?
- Consider the reflection of $\mathbb{R}^2$ about the line $y = 7x$. What are the eigenvalues (of the standard matrix)?
- Consider the $\pi/2$ rotation of $\mathbb{R}^3$ about the z-axis. What are the eigenvalues (of the standard matrix)?



0 is an eigenval of A :

$$Av = 0 \cdot v \quad v \neq 0.$$

v is nonzero vector in Nul(A).

$\Longleftrightarrow$ A not invertible.

Section 5.2

The characteristic polynomial

The eigenrecipe

Say you are given a square matrix A.

Step 1. Find the eigenvalues of A by solving

$$\det(A - \lambda I) = 0$$

Step 2. For each eigenvalue λ_i the λ_i-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

To find a basis, find the vector parametric solution, as usual.

3×3 matrices

The 3×3 case is harder. There is a version of the quadratic formula for cubic polynomials, called Cardano's formula. But it is more complicated. It looks something like this:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}.$$

There is an even more complicated formula for quartic polynomials.

One of the most celebrated theorems in math, the Abel–Ruffini theorem, says that there is no such formula for quintic polynomials.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial and eigenvalues.

$$\begin{pmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{pmatrix}$$

Characteristic polynomial: $-\lambda^3 + 9\lambda^2 - 23\lambda + 15$

This time we don't know any of the roots! We can use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

So we plug in ± 1, ± 3, ± 5, ± 15 into the polynomial and hope for the best. Luckily we find that 1, 3, and 5 are all roots, so we found all the eigenvalues!

If we were less lucky and found only one eigenvalue, we could again use long division like on the last slide.

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Typical Exam Questions 5.2

- True or false: Every $n \times n$ matrix has an eigenvalue.
- True or false: Every $n \times n$ matrix has n distinct eigenvalues.
- True or false: The nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the $n \times n$ zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues, and then factor out a linear term.

Section 5.4

Diagonalization

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if we can write:

$$A = CDC^{-1} \qquad D = \text{diagonal}$$

We say that A is similar to D.

How does this factorization of A help describe what A does to $\mathbb{R}^n$? How does this help us take powers of A?

Understanding the rabbit example: since 2 is the largest eigenvalue, (almost) all other vectors get pulled towards that eigenvector. Compare with the example from the last slide.

Diagonalization

The recipe

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

$$= \qquad C \qquad\qquad D \qquad\qquad C^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (in order). with multiplicity.

Why?

Many ways to change the diagonalization.

(1) switch order in D and C.

(2) change basis for eigenspace.

Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

More rabbits

Which ones are diagonalizable?

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hint: the characteristic polynomials are $-\lambda^3 + 3\lambda + 2$ *and* $-\lambda^3 + 2\lambda + 1$ *and both have rational roots.*

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Recipe for Diagonalization

① Find eigenvalues with alg multiplicities

e.g. $(5-\lambda)(3-\lambda)^2$ — 5 has alg mult 1, 3 has alg mult 2.

Check if sum of alg mult's is n

② For eigenval, check for eigenvals with alg. mult > 1,

if geom mult = alg mult. (always ≤) — geom mult → dim λ-eigensp = dim Nul $(A-\lambda I)$

If yes for all eigenvals ⇝ diag'able

Same as: sum of geom mult's $= n$.

Otherwise ⇝ not diag'able.

Section 5.5

Complex Eigenvalues

Three shortcuts for complex eigenvectors

Suppose we have a 2×2 matrix with complex eigenvalue λ.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.

Trace and determinant

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Now that we have complex eigenvalues, we have the following fact.

Fact. The sum of the eigenvalues of A is the trace of A and the product of the eigenvalues of A is the determinant. Need multiplicity.

Indeed, by the fundamental theorem of algebra, the characteristic polynomial factors as:

$$(x_1 - \lambda)(x_2 - \lambda) \cdots (x_n - \lambda).$$

From this we see that the product of the eigenvalues $x_1 x_2 \cdots x_n$ is the constant term, which we said was the determinant, and the sum $x_1 + x_2 + \cdots + x_n$ is $(-1)^{n-1}$ times the λ^{n-1} term, which we said was the trace.

Summary of Section 5.5

- Complex numbers allow us to solve all polynomials completely, and find n eigenvectors for an $n \times n$ matrix
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$

Fact. Real eigenvals have real eigenvectors.

Complex eigenvals have complex eigenvectors

$(-1-\lambda)(1-\lambda)+2$

λ^2+1

$i \cdot -i = 1$

$i + -i = 0$

$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ = real vector

$\begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1-i \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1-i \\ -1 \end{pmatrix}$

Typical Exam Questions 5.5

- True/False. If v is an eigenvector for A with complex entries then $i \cdot v$ is also an eigenvector for A.
- True/False. If $(i, 1)$ is an eigenvector for A then $(i, -1)$ is also an eigenvector for A.
- If A is a 4×4 matrix with real entries, what are the possibilities for the number of non-real eigenvalues of A?
- Find the eigenvalues and eigenvectors for the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Eigenvals: $(-\lambda)^2 + 1 \rightsquigarrow \lambda^2 + 1 \rightsquigarrow \lambda = \pm i$

$\underline{\lambda = i}$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow[1]{\text{shortcut}} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \overset{\text{s.c.}}{\underset{2}{\rightsquigarrow}} \begin{pmatrix} -1 \\ i \end{pmatrix}$$

$\underline{\underline{\lambda = -i}}$

Shortcut 3

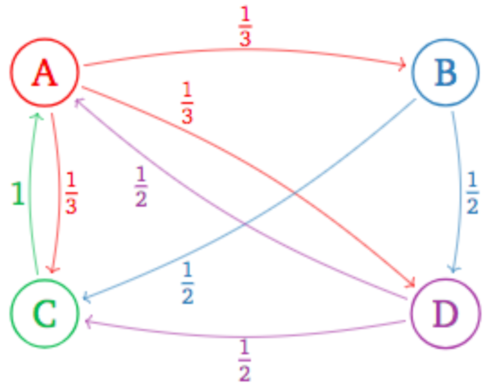
$$\overline{\begin{pmatrix} -1 \\ i \end{pmatrix}} = \begin{pmatrix} \overline{-1} \\ \overline{i} \end{pmatrix} = \begin{pmatrix} -1 \\ -i \end{pmatrix}$$

Section 5.6

Stochastic Matrices (and Google!)

Application: Web pages

Make a matrix whose ij entry is the fraction of (randomly surfing) web surfers at page j that end up at page i. If page i has N links then the ij-entry is either 0 or $1/N$.



$$\begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$$

Which web page seems most important?

Summary of Section 5.6

- A stochastic matrix is a non-negative square matrix where all of the columns add up to 1.
- Every stochastic matrix has 1 as an eigenvalue, and all other eigenvalues have absolute value at most 1.
- A positive stochastic matrix has 1-dimensional eigenspace and has a positive eigenvector. A positive 1-eigenvector with entries adding to 1 is called a steady state vector.
- For a positive stochastic matrix, all nonzero vectors approach the steady state vector under iteration.
- Steady state vectors tell us the importance of web pages (for example).

Typical Exam Questions 5.6

- Is there a stochastic matrix where the 1-eigenspace has dimension greater than 1?
- Find the steady state vector for this matrix:

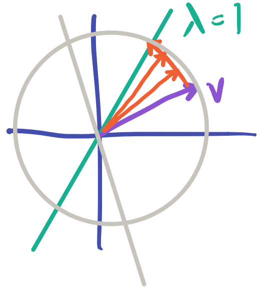
$$A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 1/3 \end{pmatrix}$$

  To what vector does $A^n \left(\begin{smallmatrix} 5 \\ 7 \end{smallmatrix}\right)$ approach as $n \to \infty$?
- Find the steady state vector for this matrix:

$$A = \begin{pmatrix} 1/3 & 1/5 & 1/4 \\ 1/3 & 2/5 & 1/2 \\ 1/3 & 2/5 & 1/4 \end{pmatrix}$$

- Make your own internet and see if you can guess which web page is the most important. Check your answer using the method described in this section.

$$A = \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}$$



Find steady state vector.

$$\begin{pmatrix} -1/2 & 1/3 \\ 1/2 & -1/3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}$$

To what vector does $A^n \begin{pmatrix} 7 \\ 8 \end{pmatrix}$ converge as $n \to \infty$

$$(8+7) \cdot \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 5.3 \\ 11.7 \end{pmatrix} \qquad \begin{pmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{pmatrix}\begin{pmatrix} 5.3 \\ 11.7 \end{pmatrix} = \text{closer to } \begin{pmatrix} 6 \\ 9 \end{pmatrix}$$