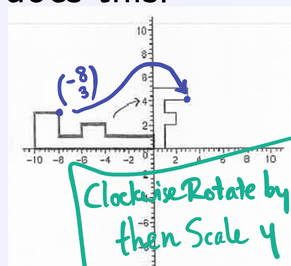
Discussion

Check. $\begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix} \begin{pmatrix} -8 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Discussion Question

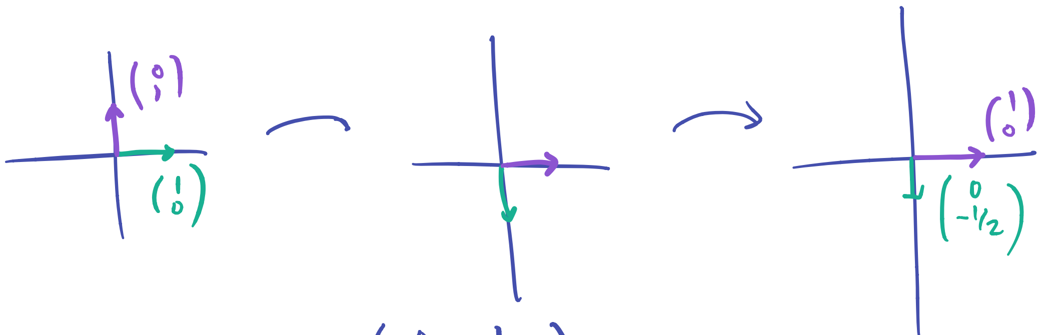
Find a matrix that does this.



Clockwise Rotate by $+\pi/2$ then Scale y by $1/2$

or

Scale x by $1/2$ Rotate by $-\pi/2$



$$\begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$$

Announcements Oct 5

- WeBWorK on Sections 3.2, 3.3 due Thursday night
- Quiz on Sections 3.2, 3.3 Friday 8 am - 8 pm EDT
- My Office Hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Second Midterm Friday Oct 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Section 3.4

Matrix Multiplication

Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things

shear right   shear down   shear right

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ clockwise rotation by $\pi/2$

Function composition

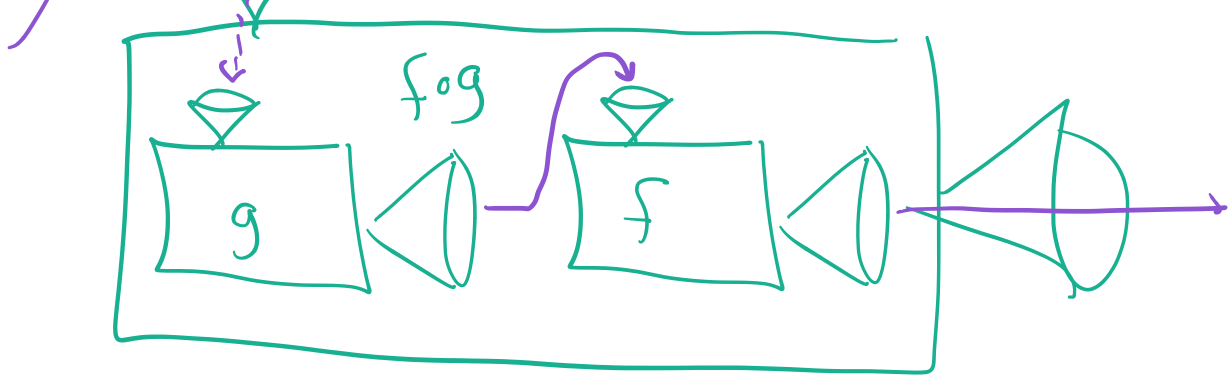
Remember from calculus that if f and g are functions then the composition $f \circ g$ is a new function defined as follows:

$$f \circ g(x) = f(g(x))$$

In words: first apply g, then f.

Example: $f(x) = x^2$ and $g(x) = x + 1$. $f \circ g(x) = (x+1)^2$ $g \circ f(x) = x^2 + 1$

Note that $f \circ g$ is usually different from $g \circ f$.



Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^p \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^p$ and make the composition $T \circ U$.

Notice that both have an p. Why?

What are the domain and codomain for $T \circ U$?

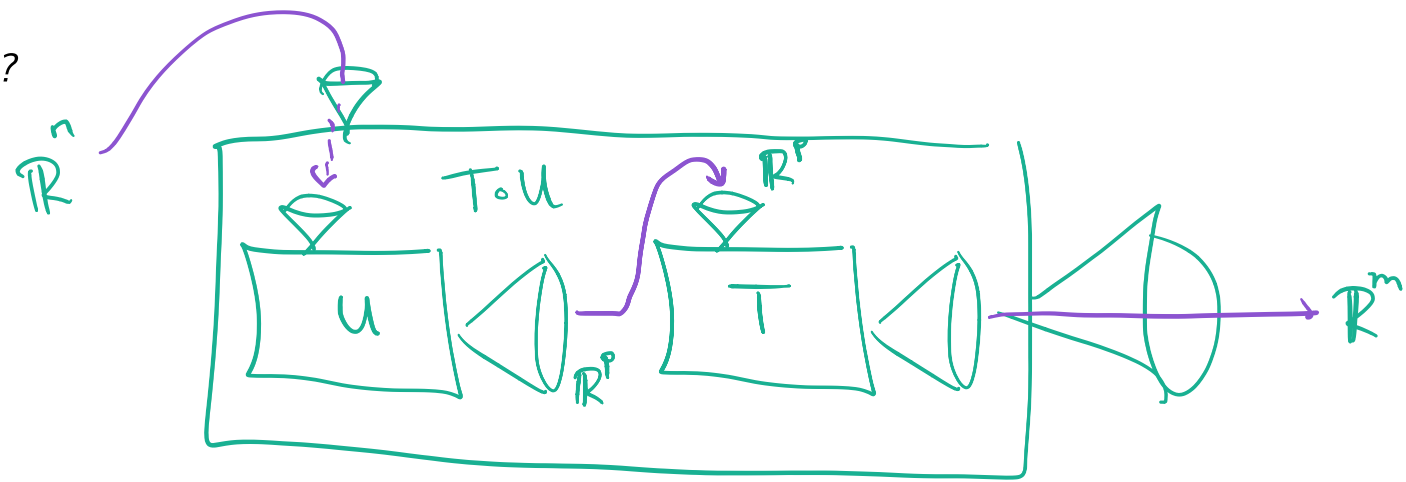
domain : $\mathbb{R}^n$

codomain : $\mathbb{R}^m$

Natural question: What is the matrix for $T \circ U$? We'll see!

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?



Composition of linear transformations

Example. $T =$ projection to y-axis and
$U =$ reflection about $y = x$ in $\mathbb{R}^2$

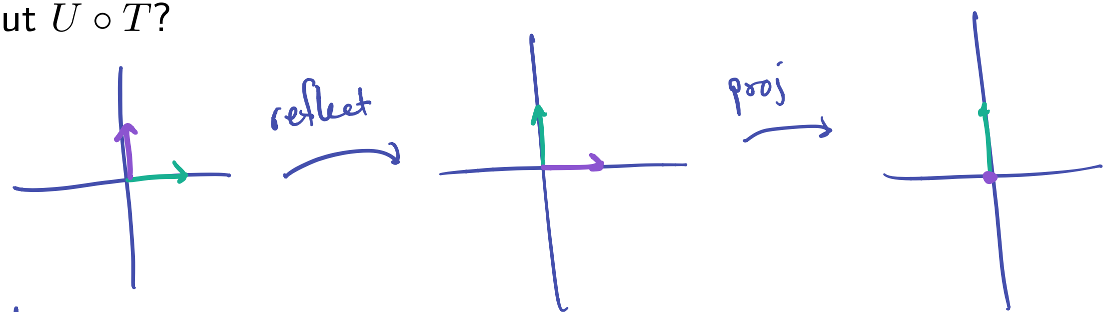
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B$

What is the standard matrix for $T \circ U$?

What about $U \circ T$?

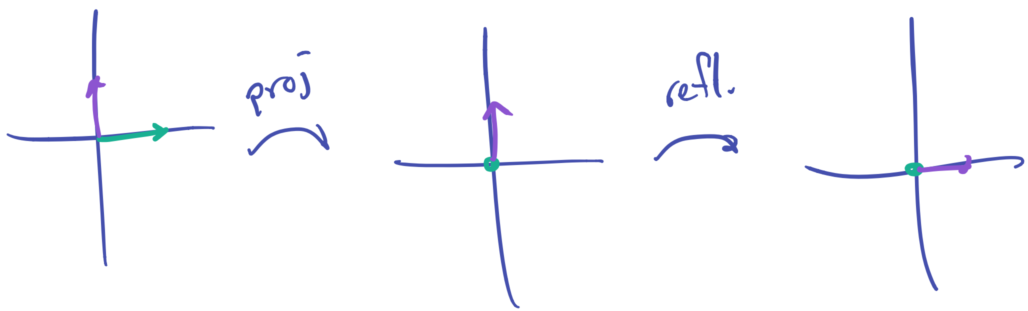
$T \circ U$



$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\neq$ not equal

$U \circ T$



$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Goal: These are AB & BA

Matrix Multiplication

And now for something completely different (not really!)

Suppose A is an $m \times n$ matrix. We write a_{ij} or A_{ij} for the ijth entry.

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where r_i is the ith row of A, and b_j is the jth column of B.

Or: the jth column of AB is A times the jth column of B.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -13 \end{pmatrix}$$

Multiply these matrices (both ways):

$$\underset{2 \times 3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \underset{3 \times 2}{\begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{pmatrix}} = \underset{2 \times 2}{\begin{pmatrix} 8 & -4 \\ 17 & -13 \end{pmatrix}}$$

Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do U then do T

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

$\mathbb{R}^p \to \mathbb{R}^m$

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case Bv is the ith column of B. So the left-hand side is A times the ith column of B. The right-hand side is the ith column of AB which we already said was A times the ith column of B. It works!

Matrix Multiplication and Linear Transformations

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

Example. $T =$ projection to y-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = A$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B$$

Already figured out

Matrix for $T \circ U$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Matrix for $U \circ T$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$ ☺

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ ☺

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that $\underline{\text{reflects}}$ through the xy-plane and then $\underline{\text{projects}}$ onto the yz-plane.

U

T

$(x,y,z) \to (0,y,z)$

$T \circ U(v) = T(U(v))$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad \text{matrix for } T$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{matrix for } U$$

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{matrix for } T \circ U$$

do second

do first

Discussion Question

Are there nonzero matrices A and B with $AB = 0$?

1. Yes
2. No

Properties of Matrix Multiplication

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$ $\quad r = \text{scalar.}$
- $\cancel{(AB)^T = B^T A^T}$
- $I_m A = A = AI_n$, where I_k is the $k \times k$ identity matrix.

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- AB is not always equal to BA $\quad$ Saw already
- $AB = AC$ does not mean that $B = C$
- $\underline{AB = 0}$ does not mean that A or B is 0

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{proj to } x} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{proj } y} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

More rabbits

Recall that the following matrix describes the change in our rabbit population from this year to the next:

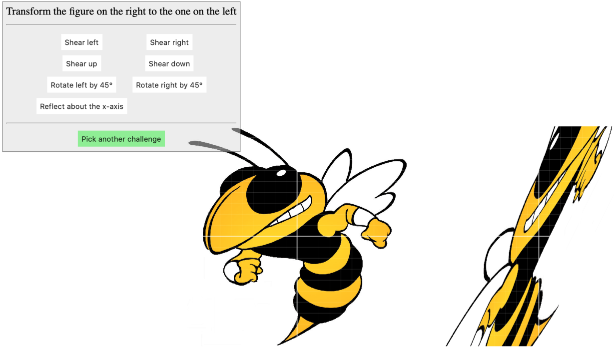
$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

What matrix should we use if we want to describe the change in the rabbit population from this year to two years from now? Or 10 years from now?

Fun with matrix multiplication

Play the Buzz game!

`http://textbooks.math.gatech.edu/ila/demos/transform_game.html`



In the rotation game, you need to find a composition of shears that gives a rotation!

Summary of Section 3.4

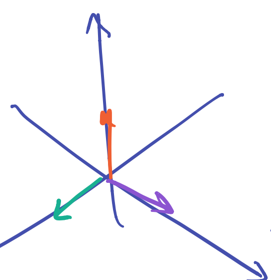
- Composition: $(T \circ U)(v) = T(U(v))$ (do U then T)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the ith column of AB is $A(b_i)$
- Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.
- Warning!
  - AB is not always equal to BA
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that A or B is 0

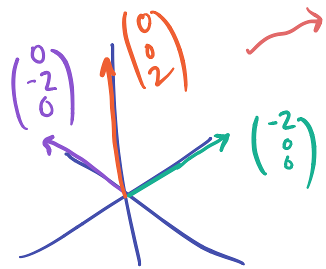
$$T\begin{pmatrix}1\\0\end{pmatrix} = T\left(\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}\right) = T\begin{pmatrix}2\\1\end{pmatrix} - T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix} - \begin{pmatrix}3\\3\\1\end{pmatrix}$$

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\1\end{pmatrix}$$

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix} - \begin{pmatrix}3\\3\\1\end{pmatrix} = \begin{pmatrix}0\\-2\\0\end{pmatrix}$$





$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}z\\0\\x\end{pmatrix}$$

$$\begin{pmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}-2 & 0 & 0\\0 & -2 & 0\\0 & 0 & 2\end{pmatrix}$$

Typical Exam Questions 3.4

- True/False. If A is a 3×4 matrix and B is a 4×3 matrix, then it makes sense to multiply A and B in both orders.
- True/False. If it makes sense to multiply a matrix A by itself, then A must be a square matrix.
- True/False. If A is a non-zero square matrix, then A^2 is a non-zero square matrix.
- True/False. If $A = -I_n$ and B is an $n \times n$ matrix, then $AB = BA$.
- Find the standard matrices for the projections to the xy-plane and the yz-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?
- Find the standard matrix A for projection to the xy-plane in $\mathbb{R}^3$. What is A^2?
- Find the standard matrix A for reflection in the xy-plane in $\mathbb{R}^3$. Is there a matrix B so that $AB = I_3$?