More rabbits

Discussion Question

Recall that the following matrix describes the change in our rabbit population from this year to the next:

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which of the following statements are true?

1. There is a population of rabbits that will result in 0 rabbits in the following year. False? only zero population
2. There are two different populations of rabbits that result in the same population in the following year False one-to-one
3. For any given population of rabbits, we can choose a population of rabbits for the current year that results in the given population in the following year (this is tricky!). ~~True? Onto?~~

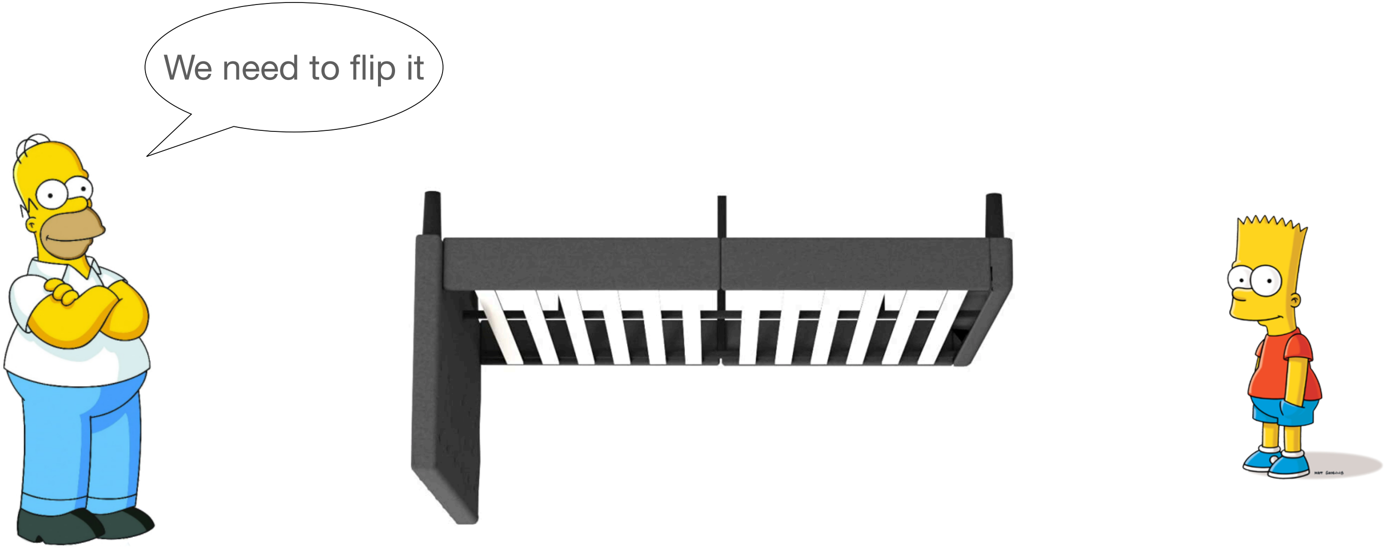
False: Fractional rabbits
Negative rabbits

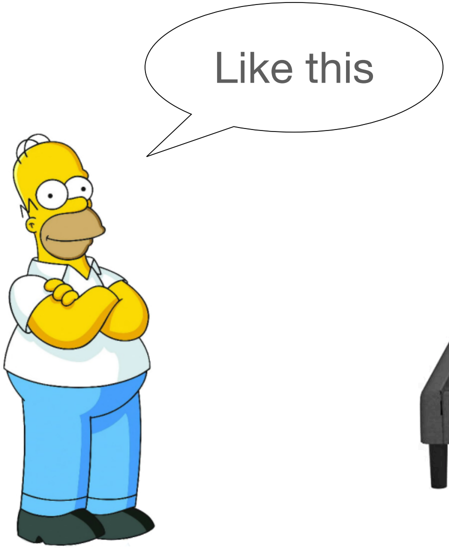
Announcements Oct 12

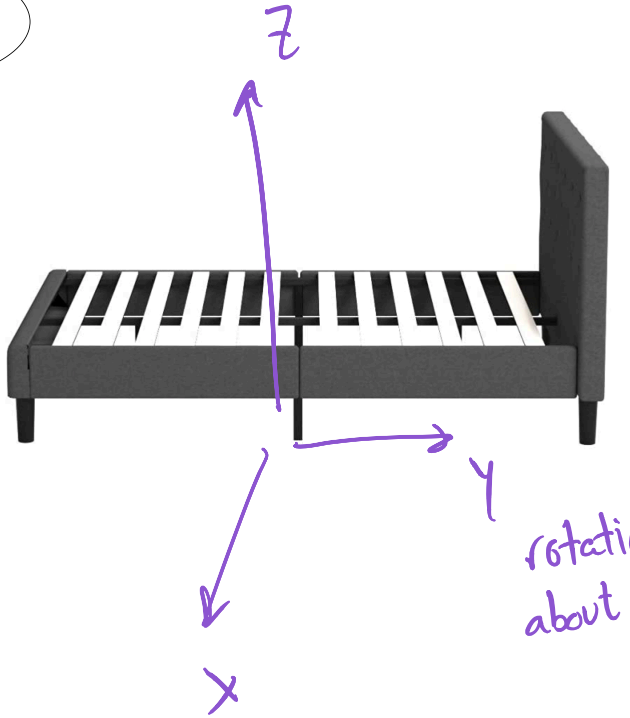
- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on Sections 3.4, 3.5, 3.6 due Thursday night
- Second Midterm Friday Oct 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- My Office Hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Review session TBA
- No studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Practice exam on Canvas

Flipping my bed

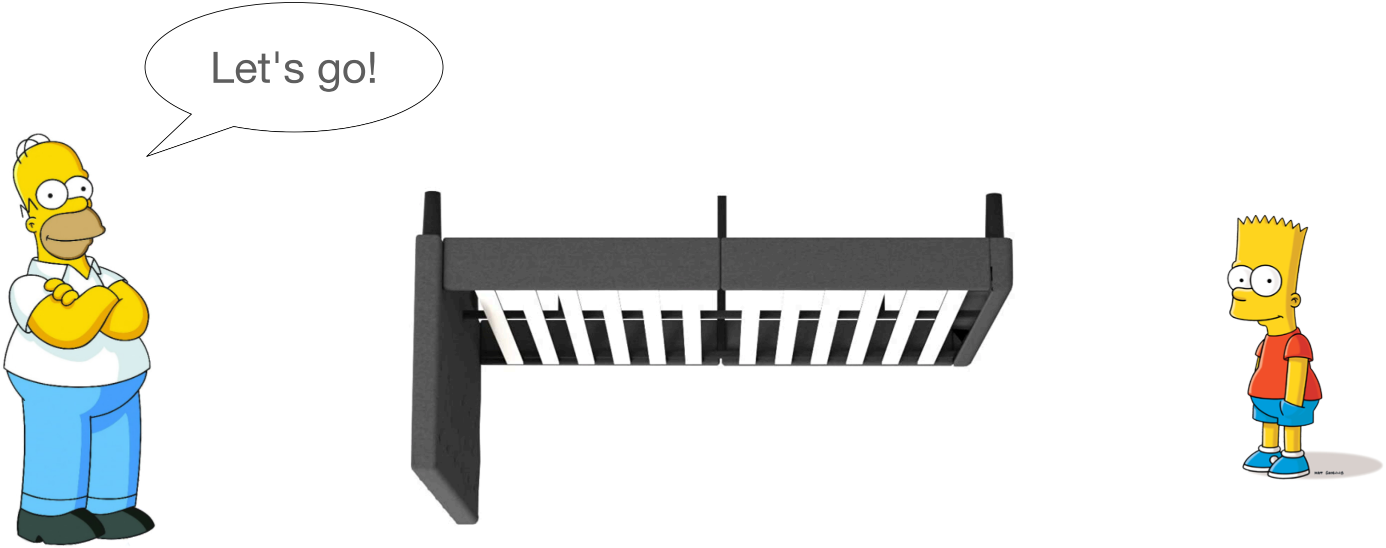

We need to flip it

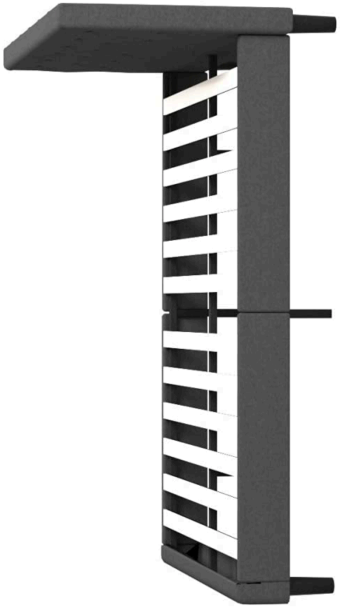




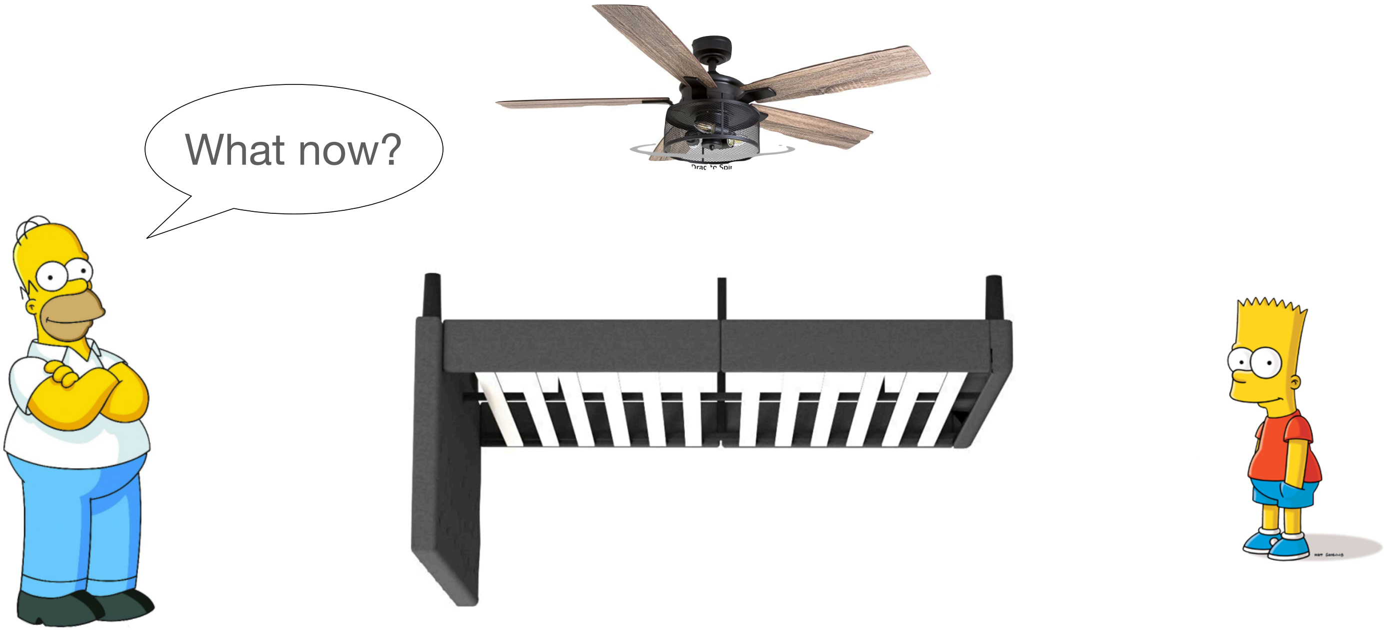
rotation by π about x-axis

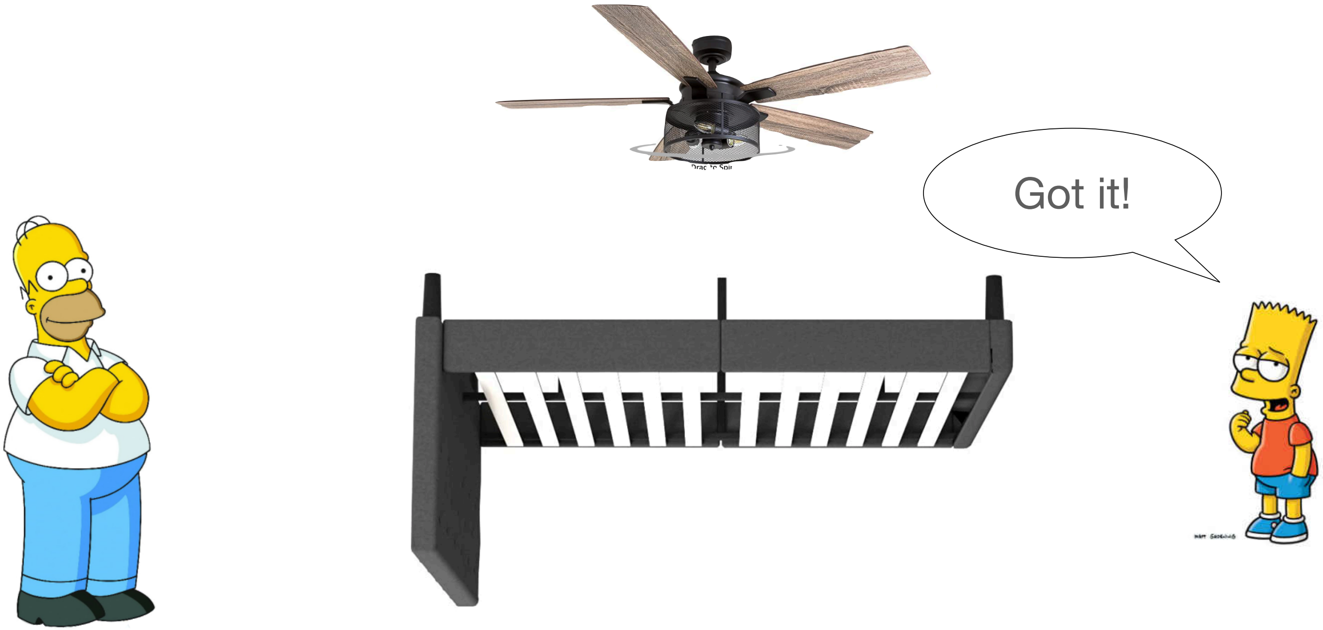
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

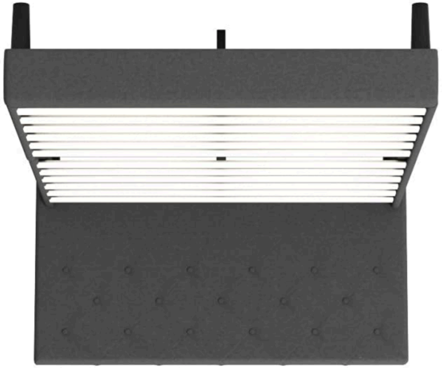

Let's go!

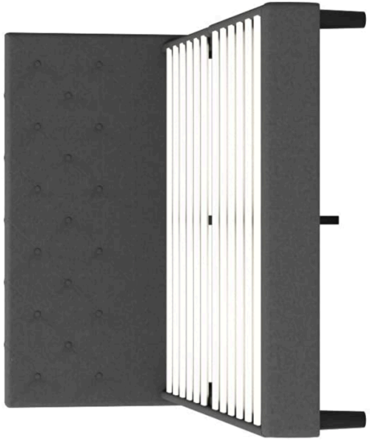



Doh!

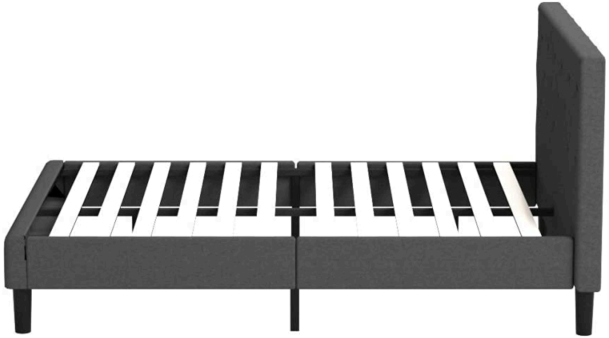

What now?


Got it!

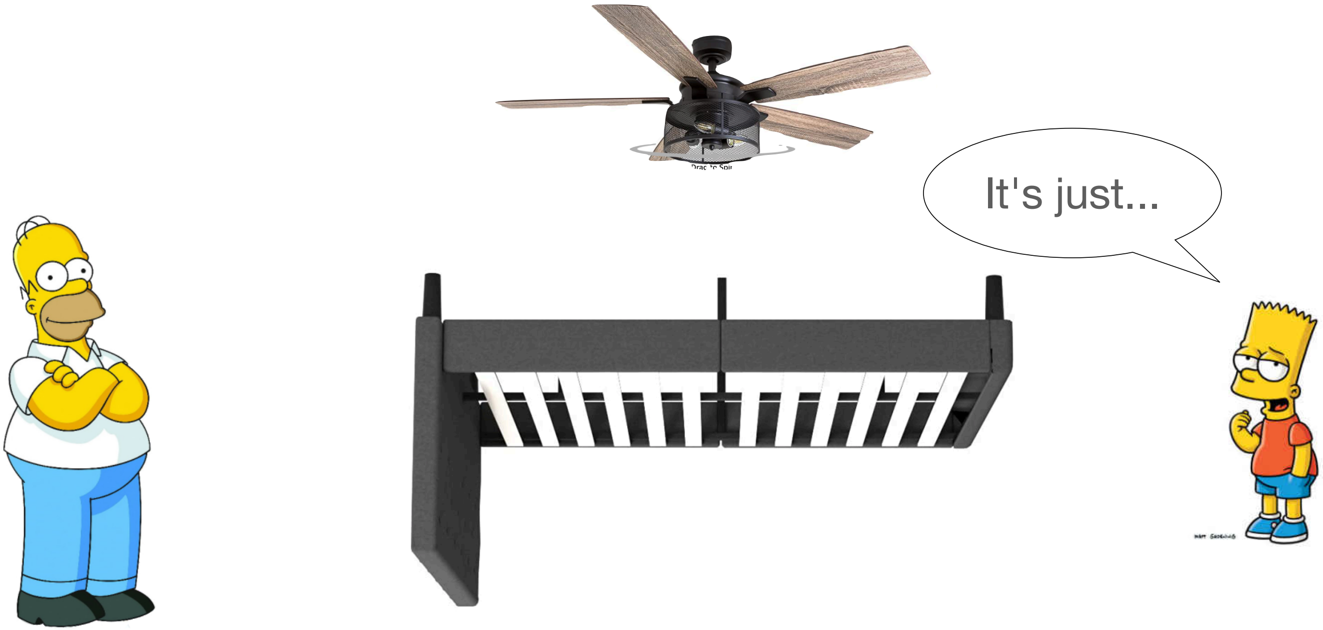


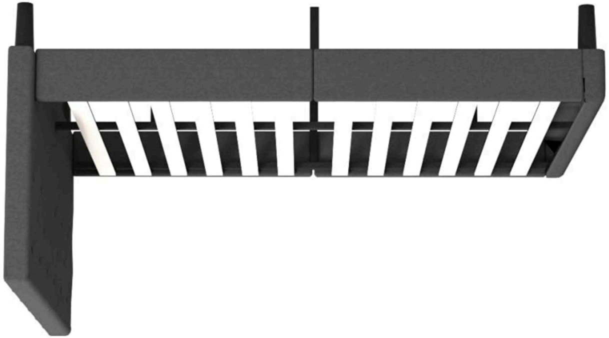


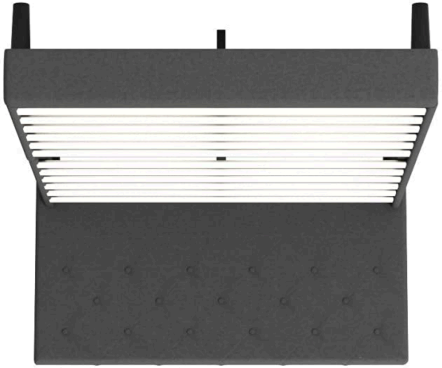


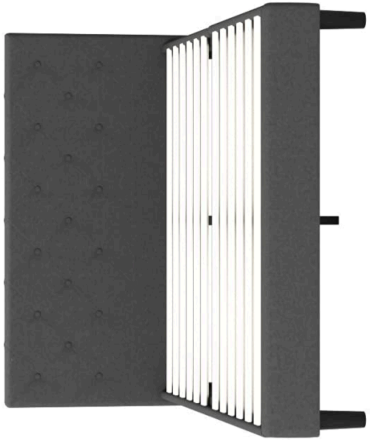

Huh?


It's just...





$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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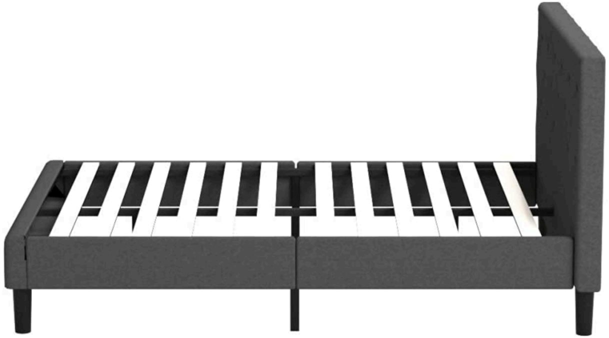

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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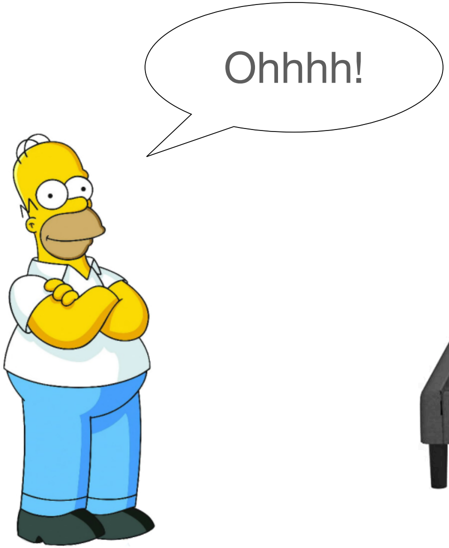
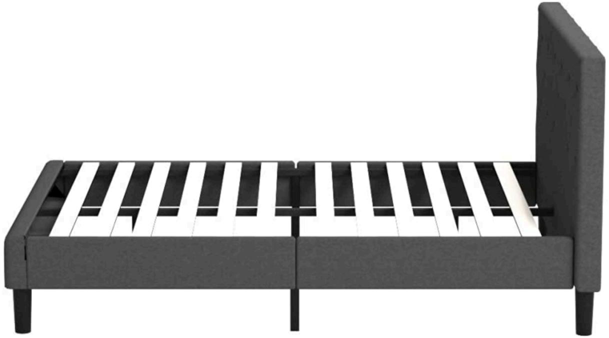
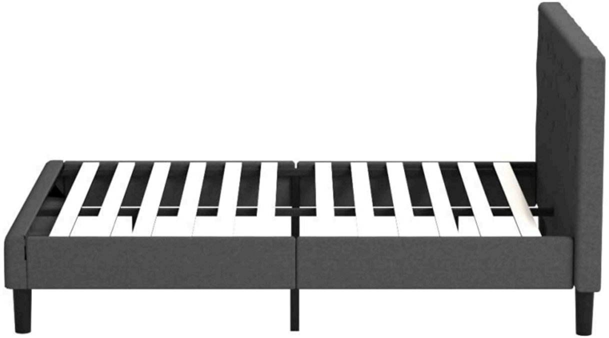




$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



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Chapter 4

Determinants

Where are we?

- We have studied the problem $Ax = b$
- We next want to study $Ax = \lambda x$
- At the end of the course we want to almost solve $Ax = b$

We need determinants for the second item.

Section 4.1

The definition of the determinant

Sec 4.2

Cofactor expansion: a formula for the determinant like ad-bc.

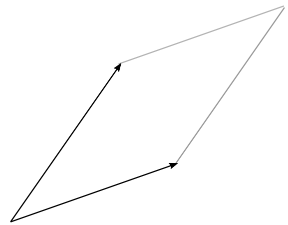
Outline of Sections 4.1 and 4.3

- Volume and invertibility
- A definition of determinant in terms of row operations
- Using the definition of determinant to compute the determinant
- Determinants of products: $\det(AB)$
- Determinants and linear transformations and volumes

Invertibility and volume

When is a 2×2 matrix invertible? Algebra

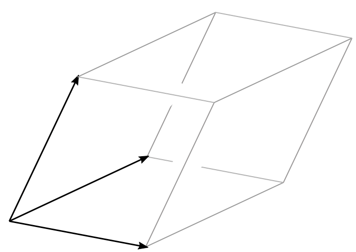
When the rows (or columns) don't lie on a line $\Leftrightarrow$ the corresponding parallelogram has non-zero area



Geometry

When is a 3×3 matrix invertible?

When the rows (or columns) don't lie on a plane $\Leftrightarrow$ the corresponding parallelepiped (3D parallelogram) has non-zero volume

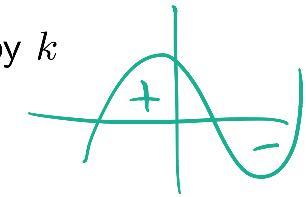


Same for $n \times n$!

The definition of determinant

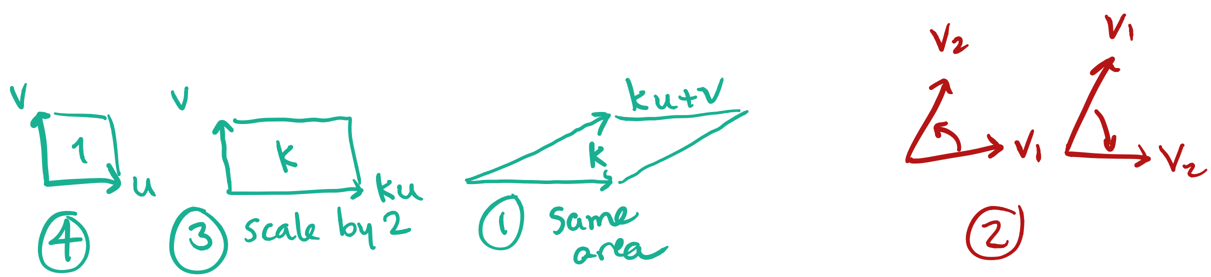
The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$



Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.



The definition of determinant

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Problem. Just using these rules, compute the determinants:

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

1   -1 17   24

use row rep.

row rep   row swap   row scale by 17

$\det = 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\det kA = k^n \det A$$

A basic fact about determinants

Fact. If A has a zero row, then $\det(A) = 0$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{row by } 0]{\text{scale bottom}} \begin{pmatrix} & \text{same} & \end{pmatrix}$$

$$\det A = 0 \cdot \det A = 0.$$

Fact. If A is a diagonal matrix then $\det(A)$ is the product of the diagonal entries.

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} = 10 \quad ✓$$

Fact. If A is in row echelon form then $\det(A)$ is the product of the diagonal entries.

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix} = 10. \quad ✓$$

Why do these follow from the definition?

A first formula for the determinant

Fact. Suppose we row reduce A. Then

$$\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$$

A — scales, swaps, replace. → row reduced A ← det =

Use the fact to get a formula for the determinant of any 2×2 matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow[\text{repl.}]{\text{row}} \begin{pmatrix} a & b \\ 0 & d - \frac{c}{a} \cdot b \end{pmatrix}$$

Scalings: $\emptyset$

Swaps: $\emptyset$

$\det = ad - bc$ (Answer.)

$\det = ad - bc$

Consequence of the above fact:

Fact. $\det A \neq 0 \Leftrightarrow A$ invertible

Numerator = 0 exactly when A is not invertible.

Computing determinants

...using the definition in terms of row operations

$$\det\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} = (-1)^{1}\,\frac{1\cdot 1\cdot -9}{1} = 9$$

$$\frac{\text{Row scales}}{\emptyset} \qquad \frac{\text{Row swaps}}{1}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix} \xrightarrow{\text{swap}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

$$\xrightarrow{\text{repl.}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 7 & -9 \end{pmatrix}$$

$$\xrightarrow{\text{rep}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

Computing determinants

...using the definition in terms of row operations

$$\det \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} = (-1)^2 \frac{1 \cdot 1 \cdot 8}{4} = 2$$

Scales

$2 \cdot 2$

$$\xrightarrow[\text{swaps}]{\text{2 row}} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 6 & 8 \end{pmatrix} \xrightarrow[\text{scales}]{\text{row}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 8 \end{pmatrix}$$

Swaps

$\parallel$

$$\xrightarrow[\text{repl.}]{\text{row}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

A Mathematical Conundrum

We have this definition of a determinant, and it gives us a way to compute it.

But: we don't know that such a determinant function exists.

More specifically, we haven't ruled out the possibility that two different row reductions might gives us two different answers for the determinant.

Don't worry! It is all okay.

We already gave the key idea: that determinant is just the volume of the corresponding parallelepiped. You can read the proof in the book if you want.

Fact 1. There is such a number $\det$ and it is unique.

Properties of the determinant

Fact 1. There is such a number $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions. next time!

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

~~Fact 6. $\det(A^T) = \det(A)$~~ ok, now we need to say what transpose is

Fact 7. $|\det(A)|$ is ~~signed~~ volume of the parallelepiped spanned by cols of A.

If you want the proofs, see the book. Actually Fact 1 is the hardest!

Powers

Fact 5. $\det(AB) = \det(A)\det(B)$

Use this fact to compute

$$\det\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix}^5\right) = 9^5$$

What is $\det(A^{-1})$? $1/9$

$$\det A \det A^{-1} = \det(AA^{-1})$$
$$9 \cdot 1/9 = 1$$

Powers

Poll

Suppose we know A^5 is invertible. Is A invertible?

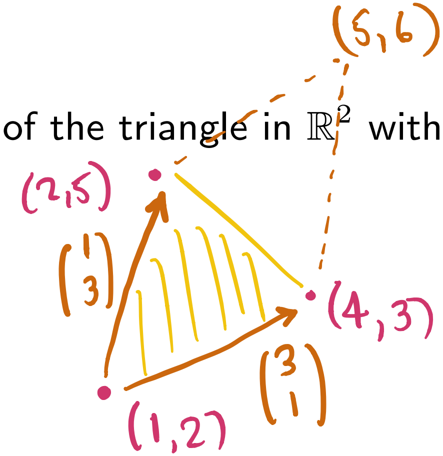
1. yes
2. no
3. maybe

Section 4.3

The determinant and volumes

Areas of triangles

What is the area of the triangle in $\mathbb{R}^2$ with vertices $(1,2)$, $(4,3)$, and $(2,5)$?



Area of parallelogram =

$$\left|\det\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}\right| = |-8| = 8$$

Area of triangle $= 8/2 = 4$

What is the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1,2)$, $(4,3)$, $(2,5)$, and $(5,6)$?

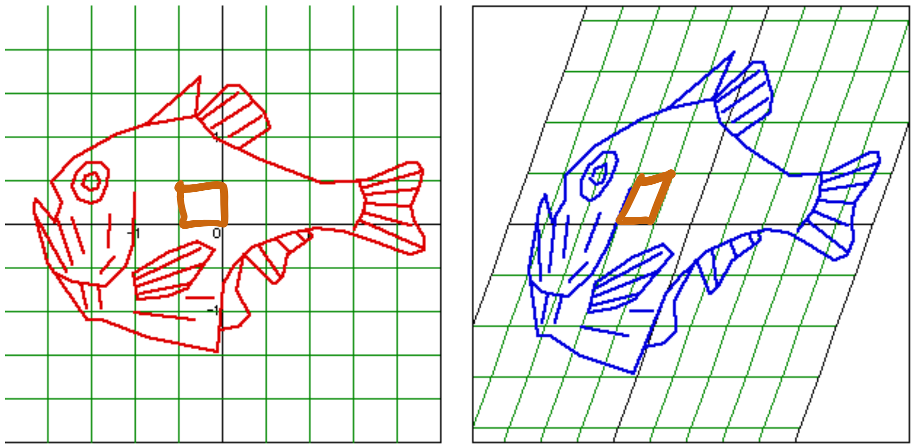
8

Determinants and linear transformations

Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

Summary of Sections 4.1 and 4.3

Say $\det$ is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

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Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

Typical Exam Questions 4.1 and 4.3

- Find the value of h that makes the determinant 0:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & h \end{pmatrix}$$

- If the matrix on the left has determinant 5, what is the determinant of the matrix on the right?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{pmatrix}$$

- If the area of a fish (in a photo) is 7 square inches, and we apply a shear, what is the new area?
- Suppose that T is a linear transformation with the property that $T \circ T = T$. What is the determinant of the standard matrix for T?
- Suppose that T is a linear transformation with the property that $T \circ T = \text{identity}$. What is the determinant of the standard matrix for T?