Announcements Oct 14

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on Sections 3.4, 3.5, 3.6 due Thursday night
- Second Midterm Friday Oct 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- My Office Hours Tue 11-12, Thu 9-10, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Review session Thu 4-5 with Seokbin
- No studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Vote now in Polls channel on Teams!

Ask questions on Piazza or in chat.

Section 2.6

Subspaces

All the ways

Here are all the ways we know to describe a subspace:

1. As span:

$$\mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

2. As a column space:

$$\mathrm{Col}\begin{pmatrix}-1 & -1\\1 & 0\\0 & 1\end{pmatrix}$$

same

3. As a null space:

$$\mathrm{Nul}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$$

$x = -y-z$
$y = y$
$z = z$

$$\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

4. As the set of solutions to a homogeneous linear system:

$$x+y+z=0$$

5. Same, but in set builder notation:

$$\left\{\begin{pmatrix}a\\b\\c\end{pmatrix} : a+b+c=0\right\}$$

same.

Section 2.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u+v$ is also in V.
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.
- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- Four things are the same: subspaces, spans, planes through 0, null spaces

Typical exam questions

- Consider the set $\{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$. Is it a subspace? If not, which properties does it fail?
- Consider the x-axis in $\mathbb{R}^3$. Is it a subspace? If not, which properties does it fail?
- Consider the set $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$. Is it a subspace? If not, which properties does it fail?
- Find spanning sets for the column space and the null space of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- True/False: The set of solutions to a matrix equation is always a subspace.
- True/False: The zero vector is a subspace.

Section 2.7

Bases

Basis theorem

Basis Theorem

If V is a k-dimensional subspace of $\mathbb{R}^n$, then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V, linearly independent, k vectors

We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

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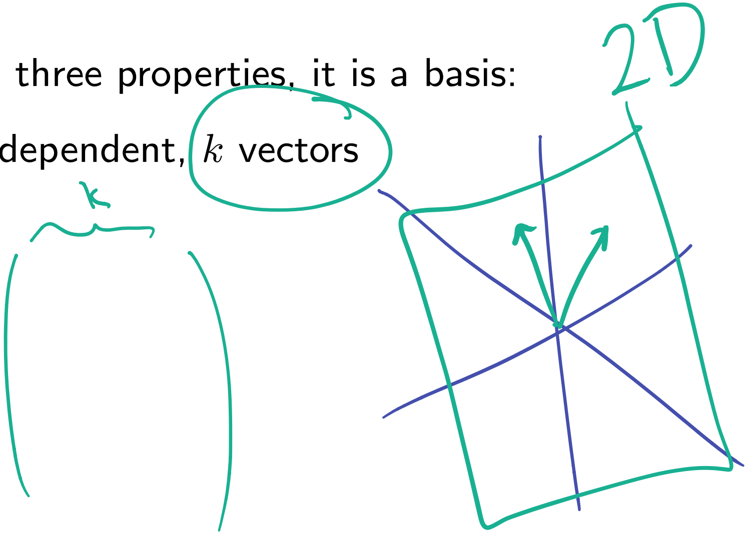


We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

Typical exam questions

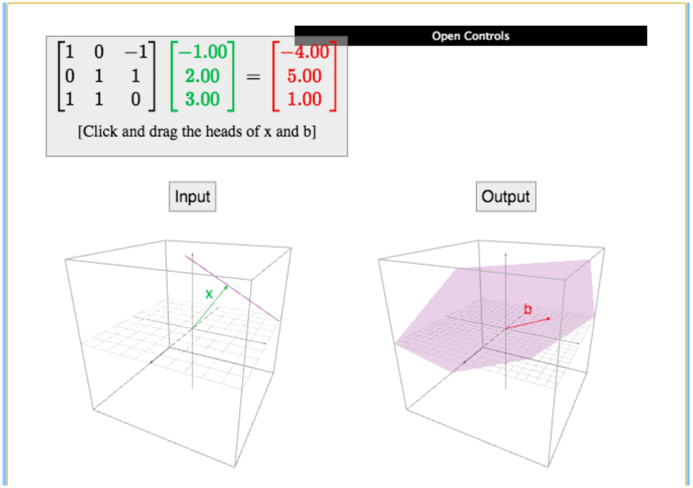
- Find a basis for the yz-plane in $\mathbb{R}^3$
- Find a basis for $\mathbb{R}^3$ where no vector has a zero
- How many vectors are there in a basis for a line in R^7?
- True/false: every basis for a plane in $\mathbb{R}^3$ has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in $\mathbb{R}^3$ and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of A is the number of pivots of A.
- True/false: If b lies in the column space of A, and the columns of A are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span R^3 must be linearly independent.

Section 2.9

The rank theorem

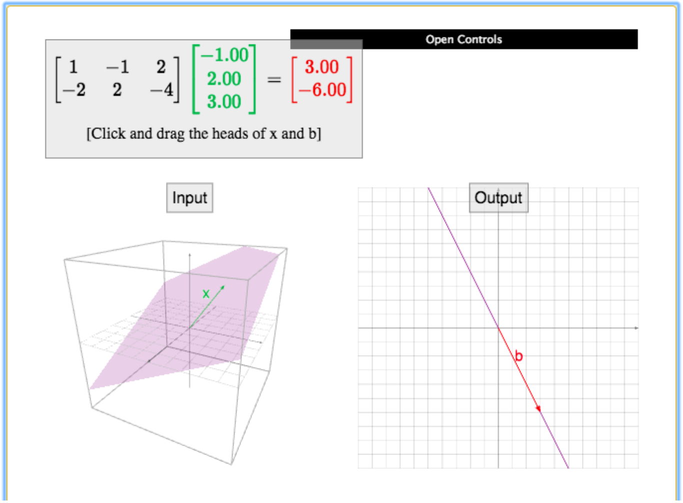
Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\mathrm{Col}(A)$:



$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1.00 \\ 2.00 \\ 3.00 \end{bmatrix} = \begin{bmatrix} -4.00 \\ 5.00 \\ 1.00 \end{bmatrix}$$

[Click and drag the heads of x and b]



$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \begin{bmatrix} -1.00 \\ 2.00 \\ 3.00 \end{bmatrix} = \begin{bmatrix} 3.00 \\ -6.00 \end{bmatrix}$$

[Click and drag the heads of x and b]

Typical exam questions

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?

Chapter 3

Linear Transformations and Matrix Algebra

Sections 3.1

Matrix Transformations

Where are we?

In Chapter 1 we learned to solve all linear systems algebraically.

In Chapter 2 we learned to think about the solutions geometrically.

In Chapter 3 we continue with the algebraic abstraction. We learn to think about solving linear systems in terms of inputs and outputs. This is similar to control systems in AE, objects in computer programming, or hot pockets in a microwave.

More specifically, we think of a matrix as giving rise to a function with inputs and outputs. Solving a linear system means finding an input that produces a desired output. We will see that sometimes these functions are invertible, which means that you can reverse the function, inputting the outputs and outputting the inputs.

The invertible matrix theorem is the highlight of the chapter; it tells us when we can reverse the function. As we will see, it ties together everything in the course.

Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

Typical exam questions

- What does the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ do to $\mathbb{R}^3$?
- What does the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- True/false. If A is a matrix and T is the associated matrix transformation, then the statement $Ax = b$ is consistent is equivalent to the statement that b is in the range of T.
- True/false. There is a matrix A so that the domain of the associated matrix transformation is a line in $\mathbb{R}^3$.

Section 3.2

One-to-one and onto transformations

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix $m \times n = A$. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are linearly independent
  - $Ax = 0$ has only the trivial solution
  - A has a pivot in each column ⟷ n pivots
  - the range has dimension n

$\dim \underbrace{\mathrm{Col}(A)}_{\text{range}} = \#\text{pivots}$

〃
n

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in each row ⟷ m pivots
  - $Ax = b$ is consistent for all b in $\mathbb{R}^m$.
  - the range of T has dimension m

<u>Not one-to-one</u>

Cols dependent

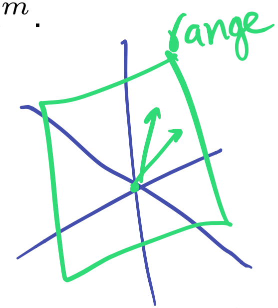
$Ax = 0$ has ∞ solns

A has $< n$ pivots

dim range $< n$

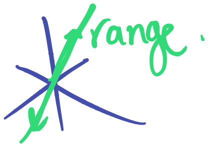
$\mathbb{R}^2 \to \mathbb{R}^3$

one-to-one



$\mathbb{R}^2 \to \mathbb{R}^3$

not one to one



Typical exam questions

- True/False. It is possible for the matrix transformation for a 5×6 matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by projection to the yz-plane is onto.
- True/False. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by π is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.

Section 3.3

Linear Transformations

Linear transformations

Which properties of a linear transformation fail for this function $T : \mathbb{R}^2 \to \mathbb{R}^2$?

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ |y| \end{pmatrix}$$

Addition: $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = T(u+v) \overset{?}{=} T(u) + T(v) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \left(\begin{matrix} 2 \\ 2 \end{matrix}\right.$$

No!

Scalar: $c = -1$ $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix} = T\begin{pmatrix} -1 \\ -1 \end{pmatrix} = T(cv) \overset{?}{=} cT(v) = -1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$ No!

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ \sin y \end{pmatrix}$$

Almost any c, u, v will work!

Both fail!

Summary of Section 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.

Every matrix transf is linear transf:

$$A(v+w) = Av + Aw$$

$$A(cv) = cAv$$

Every lin transf is a matrix transf:

Why? Recipe

$$T \rightsquigarrow A = \big(T(e_1) \cdots T(e_n)\big)$$

Recipe:

linear transf $\rightsquigarrow$ std matrix

by setup, this matrix transf. does right thing to $e_1, \ldots, e_n$

Typical Exam Questions Section 3.3

- Is the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?
- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and that

  $$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

  What is

  $$T\begin{pmatrix} 1 \\ 0 \end{pmatrix}?$$

- Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates about the z-axis by π and then scales by 2.
- Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by:

  $$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$$

  Is this a linear transformation? If so, what is the standard matrix for T?
- Is the identity transformation one-to-one?

Say $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\1\end{pmatrix}$ $T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$

Find the std matrix for T.

Need to find $T(e_1)$ & $T(e_2)$

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\left(\begin{pmatrix}2\\1\end{pmatrix}-\begin{pmatrix}1\\1\end{pmatrix}\right) = T\begin{pmatrix}2\\1\end{pmatrix} - T\begin{pmatrix}1\\1\end{pmatrix}$$

$$= \begin{pmatrix}3\\1\\1\end{pmatrix} - \begin{pmatrix}3\\3\\1\end{pmatrix} = \begin{pmatrix}0\\-2\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \left(2\cdot\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}2\\1\end{pmatrix}\right)$$

$$= 2\cdot\begin{pmatrix}3\\3\\1\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}3\\5\\1\end{pmatrix}$$

$$\boxed{\begin{pmatrix}0 & 3\\-2 & 5\\0 & 1\end{pmatrix}}$$

Section 3.4

Matrix Multiplication

Summary of Section 3.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do U then T)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the ith column of AB is $A(b_i)$
- Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.
- Warning!
  - AB is not always equal to BA
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that A or B is 0

Typical Exam Questions 3.4

- True/False. If A is a 3×4 matrix and B is a 4×3 matrix, then it makes sense to multiply A and B in both orders.
- True/False. If it makes sense to multiply a matrix A by itself, then A must be a square matrix.
- True/False. If A is a non-zero square matrix, then A^2 is a non-zero square matrix.
- True/False. If $A = -I_n$ and B is an $n \times n$ matrix, then $AB = BA$.
- Find the standard matrices for the projections to the xy-plane and the yz-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?
- Find the standard matrix A for projection to the xy-plane in $\mathbb{R}^3$. What is A^2?
- Find the standard matrix A for reflection in the xy-plane in $\mathbb{R}^3$. Is there a matrix B so that $AB = I_3$?

Solve for X :

$$AX(D+BX)^{-1}\,{}^{\swarrow (D+BX)} = C(D+BX)$$

$$AX = C(D+BX)$$

$$AX = CD + CBX$$

$$AX - CBX = CD$$

$$(A-CB)^{-1}(A-CB)X = \overset{(A-CB)^{-1}}{\wedge}CD$$

$$X = (A-CB)^{-1}CD$$

Section 3.5

Matrix Inverses

Summary of Section 3.5

- A is invertible if there is a matrix B (called the inverse) with
$$AB = BA = I_n$$
- For a 2×2 matrix A we have that A is invertible exactly when $\det(A) \neq 0$ and in this case
$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
- If A is invertible, then $Ax = b$ has exactly one solution:
$$x = A^{-1}b.$$
- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A \,|\, I_n)$.
- Invertible linear transformations correspond to invertible matrices.

only 2x2.

~~Similar~~ formula for nxn.

Find $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{pmatrix}^{-1}$ $\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 8 & 0 & 0 & 1 \end{array}\right)$ row reduce.

Works for nxn

Typical Exam Questions 3.5

- Find the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

- Find a 2×2 matrix with no zeros that is equal to its own inverse.
- Solve for the matrix X. Assume that all matrices that arise are invertible:

$$AX(C + DX)^{-1} = B$$

- True/False. If A is invertible, then A^2 is invertible?
- Which linear transformation is the inverse of the clockwise rotation of $\mathbb{R}^2$ by $\pi/4$?
- True/False. The inverse of an invertible linear transformation must be invertible.
- Find a matrix with no zeros that is not invertible.
- Are there two different rabbit populations that will lead to the same population in the following year?

Section 3.6

The invertible matrix theorem

Main content: For a □ matrix,

pivot in each row

$\Longleftrightarrow$ pivot in each col.

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

(1) A is invertible
(2) T is invertible
(3) The reduced row echelon form of A is I_n
(4) A has n pivots
(5) $Ax = 0$ has only 0 solution
(6) $\mathrm{Nul}(A) = \{0\}$
(7) $\mathrm{nullity}(A) = 0$
(8) columns of A are linearly independent
(9) columns of A form a basis for $\mathbb{R}^n$
(10) T is one-to-one
(11) $Ax = b$ is consistent for all b in $\mathbb{R}^n$
(12) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$
(13) columns of A span $\mathbb{R}^n$
(14) $\mathrm{Col}(A) = \mathbb{R}^n$
(15) $\mathrm{rank}(A) = n$
(16) T is onto
(17) A has a left inverse
(18) A has a right inverse

col

row

transpose

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

(20) A^T invertible
(21) rows lin ind
(22) rows span $\mathbb{R}^n$
(23) rows basis $\mathbb{R}^n$

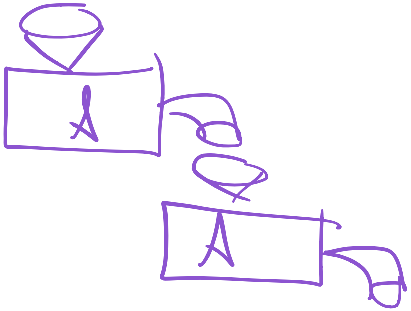
(19) $\det A \neq 0$.

The Invertible Matrix Theorem

Poll

Which are true? Why?

- m) If A is invertible then the rows of A span $\mathbb{R}^n$ yes
- n) If $Ax = b$ has exactly one solution for ~~all~~ each b in $\mathbb{R}^n$ then A is row equivalent to the identity. yes
- o) If A is invertible then A^2 is invertible yes (det)
- p) If A^2 is invertible then A is invertible yes (det)



Typical Exam Questions Section 3.6

In all questions, suppose that A is an $n \times n$ matrix and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. For each question, answer YES or NO.

(1) Suppose that the reduced row echelon form of A does not have any zero rows. Must it be true that $Ax = b$ is consistent for all b in $\mathbb{R}^n$?

(2) Suppose that T is one-to-one. Is is possible that the columns of A add up to zero?

(3) Suppose that $Ax = e_1$ is not consistent. Is it possible that T is onto?

(4) Suppose that $n = 3$ and that $T \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 0$. Is it possible that T has exactly two pivots?

(5) Suppose that $n = 3$ and that T is one-to-one. Is it possible that the range of T is a plane?

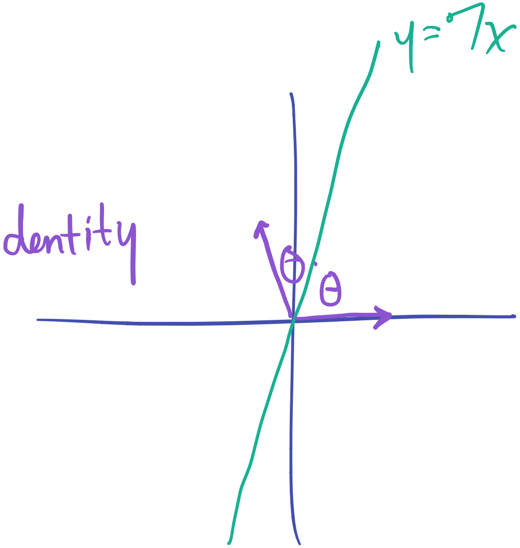
T is refl. about $y=7x$ in $\mathbb{R}^2$

$A=$ std matrix for T

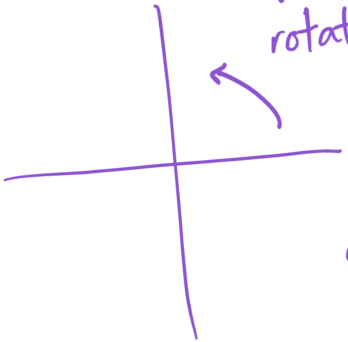
What is A^2?

A^2 is the std matrix for $T \circ T =$ identity

$\to I$



Want $A^4 = I$



$T =$ rotation by $\pi/2$

$T \circ T \circ T \circ T = \text{id}.$

So: find std matrix for T