Announcements Oct 19

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on Determinants I due Thursday night
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- My Office Hours Tue 11-12, Thu 9-10, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4

  Quiz Fri on Det I.

- Studio on Friday
- Tutoring: http://tutoring.gatech.edu/tutoring
- PLUS sessions: http://tutoring.gatech.edu/plus-sessions
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: https://counseling.gatech.edu

Chapter 4

Determinants

Where are we?

- We have studied the problem $Ax = b$
- We next want to study $Ax = \lambda x$
- At the end of the course we want to almost solve $Ax = b$
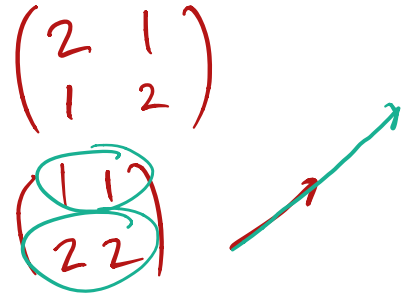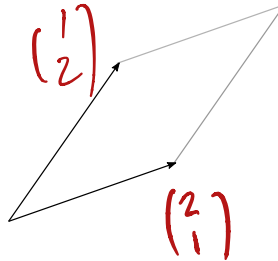
We need determinants for the second item.

Section 4.1

The definition of the determinant

Invertibility and volume

When is a 2×2 matrix invertible? $\leftarrow$ Algebra

When the rows (or columns) don't lie on a line $\Leftrightarrow$ the corresponding parallelogram has non-zero area. $\leftarrow$ Geometry

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

When is a 3×3 matrix invertible?

When the rows (or columns) don't lie on a plane $\Leftrightarrow$ the corresponding parallelepiped (3D parallelogram) has non-zero volume
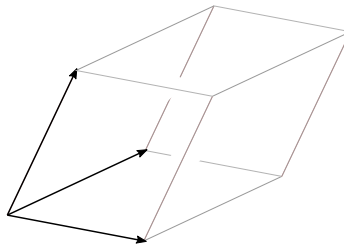
Same for $n \times n$!

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

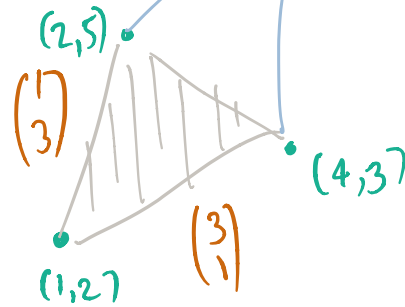Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.

Section 4.3

The determinant and volumes

Areas of triangles

What is the area of the triangle in $\mathbb{R}^2$ with vertices $(1,2)$, $(4,3)$, and $(2,5)$?

$(2,5)$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$(4,3)$

$(1,2)$

$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\left| \det \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \right| = \text{area of } \square$$

$$\parallel$$

$$8$$

area of $\triangle = 4$.

What is the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1,2)$, $(4,3)$, $(2,5)$, and $(5,6)$?
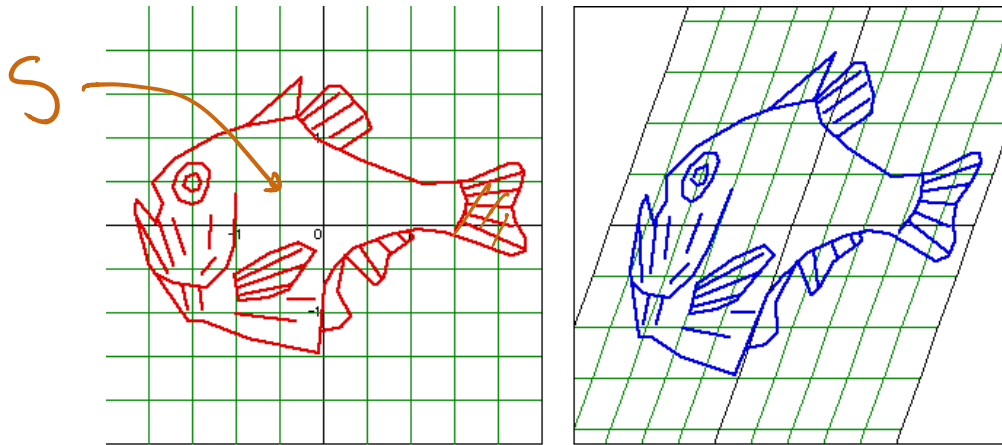
$$8$$

Determinants and linear transformations

Say A is an $n \times n$ matrix and $T(v) = Av$.

in 2D, vol means area.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:

S

Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

Example. What is area of this ellipse?

Take $S =$ unit circle area 2π

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathrm{vol}(T(S)) = 2 \cdot 2\pi$.

Summary of Sections 4.1 and 4.3

Say $\det$ is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$

2. If we do a row replacement on a matrix, the determinant is unchanged

3. If we swap two rows of a matrix, the determinant scales by -1

4. If we scale a row of a matrix by k, the determinant scales by k

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix}$$

upper triang.   lower triang

$$\begin{pmatrix} 1 & 2 \\ 7 & 19 \end{pmatrix}$$

Fact 1. There is such a function $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\dfrac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

today!

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S)$.

Typical Exam Questions 4.1 and 4.3

- Find the value of h that makes the determinant 0:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & h \end{pmatrix}$$

- If the matrix on the left has determinant 5, what is the determinant of the matrix on the right?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{pmatrix}$$

- If the area of a fish (in a photo) is 7 square inches, and we apply a shear, what is the new area?

- Suppose that T is a linear transformation with the property that $T \circ T = T$. What is the determinant of the standard matrix for T?

- Suppose that T is a linear transformation with the property that $T \circ T = \text{identity}$. What is the determinant of the standard matrix for T?

Section 4.2

Cofactor expansions

A formula for the determinant

We will give a recursive formula.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$$

First some terminology:

$A_{ij} = ij$th minor of A
$\quad = (n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column

$C_{ij} = (-1)^{i+j} \det(A_{ij})$
$\quad = ij$th cofactor of A

Above example

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$
$$= 1 \cdot -3 \; - 2 \cdot -6 \; + 3 \cdot -3$$
$$= -3 + 12 - 9 = 0 .$$

Finally:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Or:

Signs alternate

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

A formula for the determinant

For the recursive formula:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

Need to start somewhere...

1×1 matrices

$$\det(a_{11}) = a_{11}$$

2×2 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \cdot \det a_{22} - a_{12} \cdot \det a_{21}$$
$$= a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

A formula for the determinant

3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \cdots$$

You can write this out. And it is a good exercise. But you won't want to memorize it.

$$a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - \quad \smile \quad - \quad - \quad \smile \quad - \quad \smile$$

$$= \quad a_{11} \left(a_{22} a_{33} - a_{23} a_{32} \right) - \quad - \quad \diagdown \quad - \quad \diagdown$$

Determinants

Compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

alt. signs

$$= +5 \cdot \boxed{-3} - 1 \cdot \boxed{-7} + 0 \cdot \left(\begin{array}{c}\text{don't}\\\text{care}\end{array}\right)$$

$$= -8$$

Compute

$$\det \begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 7 & 1 & 0 \\ -1 & 9 & 3 & 2 \\ 4 & 11 & 0 & -1 \end{pmatrix} = 0 \cdot \left(\begin{array}{c}\text{don't}\\\text{care}\end{array}\right) - 1 \cdot \det(-8)$$

$$+ 0 \cdot \left(\begin{array}{c}\text{don't}\\\text{care}\end{array}\right) - 0 \cdot \left(\begin{array}{c}\text{don't}\\\text{care}\end{array}\right)$$

$$= +8$$

A formula for the determinant

Another formula for 3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

down & right

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

down & left

Use this formula to compute

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$-15 + 8 + 0$

$-0 - 1 - 0$

$= -8$

The formula we gave for $\det(A)$ is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in} \text{ for any fixed } i$$
$$\det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj} \text{ for any fixed } j$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 \end{pmatrix}$$

Or:

Correct for odd rows/cols

$$\det(A) = a_{i1}(\det(A_{i1})) - a_{i2}(\det(A_{i2})) + \cdots \pm a_{in}(\det(A_{in}))$$
$$\det(A) = a_{1j}(\det(A_{1j})) - a_{2j}(\det(A_{2j})) + \cdots \pm a_{nj}(\det(A_{nj}))$$

need to fix.

Compute:

$$\det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$+ 0 \cdot (\quad) - 0 \cdot (\quad) + 1 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$i j$ entry = sign of $(-1)^{i+j}$

$$= 1 \cdot 1$$
$$= 1$$

Determinants of triangular matrices

If A is upper (or lower) triangular, $\det(A)$ is easy to compute:

$$\det \begin{pmatrix} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & \boxed{+} \end{array}$$

use bot row or left col.

bottom row:

$$10 \cdot \det \begin{pmatrix} 2 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

bottom row

$$10 \cdot 5 \cdot \det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$$10 \cdot 5 \cdot 2 = 100.$$

Determinants

What is the determinant?

$$\det \begin{pmatrix} 0 & 7 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 4 & 5 & 1 & 7 \\ 8 & 9 & 0 & 10 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

cofactor expansion using 3rd col.

$$-1 \cdot \boxed{\uparrow}$$

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 8 & 9 & 10 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= -1 \cdot 0 = 0.$$

A formula for the inverse

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

divide by det.

$n \times n$ matrices

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix}$$

$$= \frac{1}{\det(A)} (C_{ij})^T$$

Check that these agree!

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$C_{11} = d \quad C_{12} = -c$

$C_{21} = -b \quad C_{22} = a$

$\rightarrow \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

transpose $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The proof uses Cramer's rule (see the notes on the course home page. We're not testing on this - it's just for your information.)

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.

Typical Exam Questions 4.2

- True or false. The cofactor expansion across the first row gives the negative of the cofactor expansion across the second row.

- Find the determinant of the following matrix using one of the formulas from this section:
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & 0 & 9 \end{pmatrix}$$

- Find the determinant of the following matrix using one of the formulas from this section:
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$

- For fun, find the inverse of the above matrix using the formula from this section.

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or}$$
> $$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), principal component analysis, Google, Netflix, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

A Question from Biology

In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year - think of it as a vector (f, s, t) - what is the population the next year?

Now choose some starting population vector $u = (f, s, t)$ and choose some number of years N. What is the new population after N years?

▶ Demo

Chapter 5

Eigenvectors and eigenvalues

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

eigen = characteristic

So Av points in the same direction as v.

This the the most important definition in the course.

▸ Demo

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

Can you find any eigenvectors/eigenvalues for the following matrix?

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

What happens when you apply larger and larger powers of A to a vector?

Eigenvectors and Eigenvalues
Examples

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix}, \quad \lambda = 2$$

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4$$

How do you check?

Poll

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
are eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?