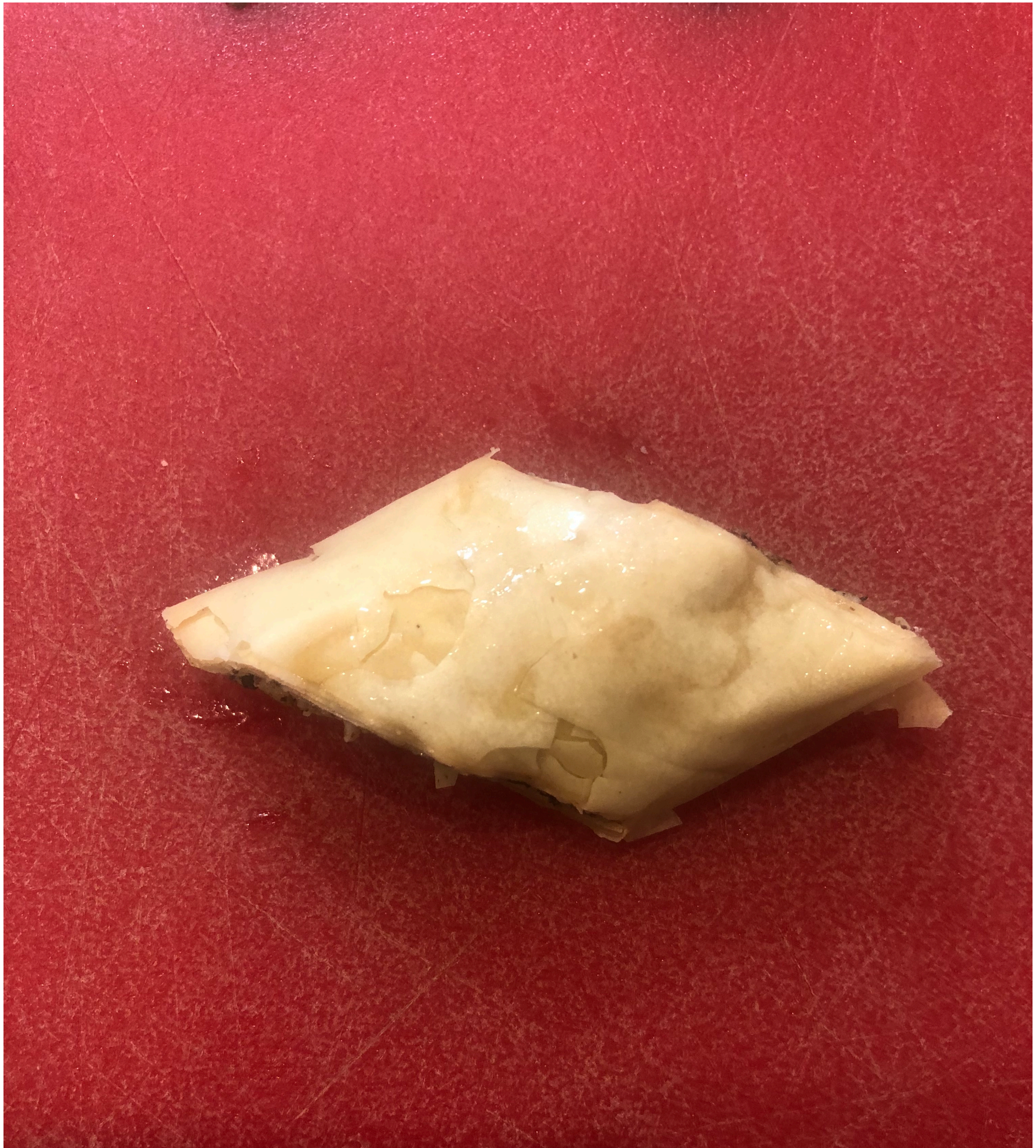
Announcements Oct 21

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on Determinants I due Thursday night
- Quiz on Sections 4.1, 4.3 Friday 8 am - 8 pm EDT
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- My Office Hours Tue 11-12, Thu ~~9-10~~ 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`



Chapter 5

Eigenvectors and eigenvalues

Where are we?

Remember:

Almost every engineering problem, no matter how huge, can be reduced to linear algebra:

$$Ax = b \quad \text{or}$$
$$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), fluid mixing, RLC circuits, clustering (data analysis), principal component analysis, Google, Netflix (collaborative prediction), infectious disease models, special relativity, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

A Question from Biology

In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year - think of it as a vector (f, s, t) - what is the population the next year?

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 6s + 8t \\ 1/2 f \\ 1/2 s \end{pmatrix}$$

eigenval ≈ 2

Now choose some starting population vector $u = (f, s, t)$ and choose some number of years N. What is the new population after N years?

eigenvector.

$A^N \begin{pmatrix} f \\ s \\ t \end{pmatrix}$

also population ~doubles each years

After many years, population approaches ratio of 16:4:1 regardless of initial population.

Demo

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

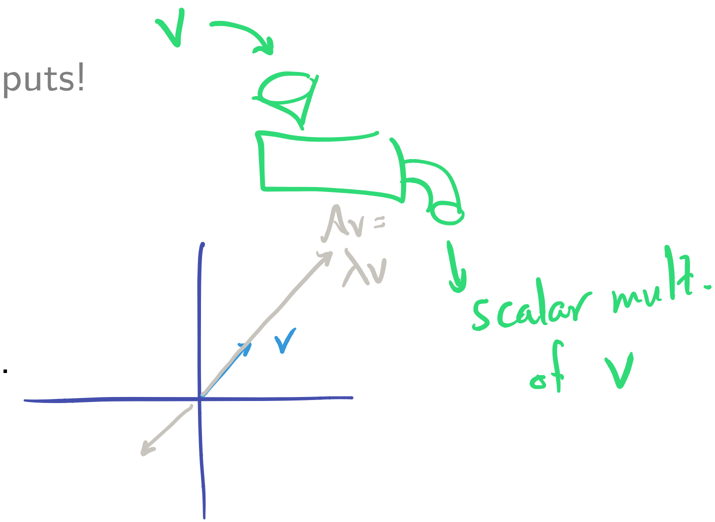
Av is a scalar mult. of v

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

Think of this in terms of inputs and outputs!

eigen $=$ *characteristic (or: self)*

So Av points in the same direction as v.



This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

0 is not an eigenvector

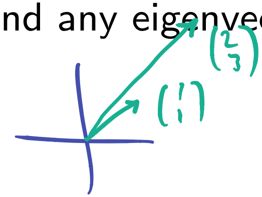
Suppose A is an $n \times n$ matrix and there is a $\boxed{v \neq 0}$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

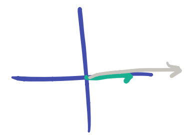
then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

or any nonzero multiple of $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

Can you find any eigenvectors/eigenvalues for the following matrix?



$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

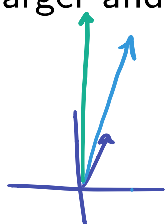


$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix}$$

So $\begin{pmatrix}1\\1\end{pmatrix}$ not eigenvector. $\begin{pmatrix}1\\0\end{pmatrix}$ is with eigenvalue 2 eigenvalue 3

What happens when you apply larger and larger powers of A to a vector?

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^n \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2^n\\3^n\end{pmatrix}$$



lengths roughly tripling each time.

$\lambda = 3$ (biggest eigenval)

Eigenvectors and Eigenvalues

Examples

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 16 \;\; 32 \\ 4 \;\; 8 \\ 1 \;\; 2 \end{pmatrix}, \quad \lambda = 2$$

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 64 \\ 16 \\ 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4$$

$$\begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

How do you check?

$$\begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = \text{not a multiple of } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

not an eigenvector.

Eigenvectors and Eigenvalues

Confirming eigenvectors

Poll

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?

Eigenvectors and Eigenvalues
Confirming eigenvalues

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 17 \\ 99 \end{pmatrix} = \begin{pmatrix} 17 \\ 99 \end{pmatrix}$

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

$Av = 3v$

Same!

$Av - 3v = 0$

$\cancel{(A-3)v = 0}$??

$Av - 3Iv = 0$

$\underline{(A - 3I)}\, v = 0$

matrix

This is an $Ax = 0$ problem!

Q. does $A - 3I$ have a nontrivial null space?

$3I$

$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$

"subtract 3 off the diag"

Answer: Yes! one pivot

$\xrightarrow{\text{row red}} \begin{pmatrix} +1 & +4 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{array}{l} x = -4y \\ y = y \end{array} \quad y\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$v =$ any nonzero! multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

What is a general procedure for finding eigenvalues?

Eigenspaces

λ-eigenvector: eigenvector with eigenvalue λ
= nonzero vector in λ-eigenspace

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

Why is this a subspace? It's a null space!

So if $\begin{pmatrix}1\\1\\1\end{pmatrix}$ & $\begin{pmatrix}1\\2\\3\end{pmatrix}$ are 7-eigenvectors then all lin combos of these are 7 eigenvectors.

Fact. λ-eigenspace for $A = \mathrm{Nul}(A - \lambda I)$

Example. Find the eigenspaces for $\lambda = 2$ and $\lambda = -1$ and sketch.

lines

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

$\lambda = 2$

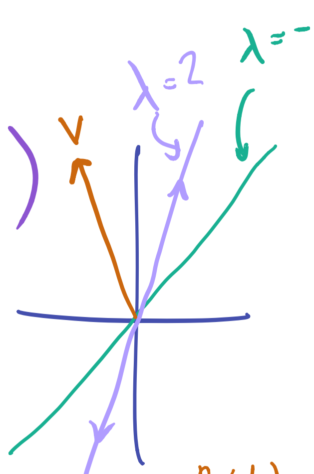
$$\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

subtract 2 off diag

$$\xrightarrow{\text{vpf}} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$\lambda = -1$

$$\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$



$v = a \cdot \begin{pmatrix}2\\1\end{pmatrix} + b \begin{pmatrix}1\\1\end{pmatrix}$

$A^n v = A^n\left(a \cdot \begin{pmatrix}2\\1\end{pmatrix} + b \cdot \begin{pmatrix}1\\1\end{pmatrix}\right) = a \cdot A^n \begin{pmatrix}2\\1\end{pmatrix} + b \cdot A^n \begin{pmatrix}1\\1\end{pmatrix} = a \cdot 2^n \begin{pmatrix}2\\1\end{pmatrix} + b \cdot (-1)^n \begin{pmatrix}1\\1\end{pmatrix}$

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

Eigenvalues

And invertibility

Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A

Why?

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Important! You can not find the eigenvalues by row reducing first! After you find the eigenvalues, you row reduce $A - \lambda I$ to find the eigenspaces. But once you start row reducing the original matrix, you change the eigenvalues.

Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.

Why?

Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

- Reflection about the line $y = -x$ in $\mathbb{R}^2$
- Orthogonal projection onto the x-axis in $\mathbb{R}^2$
- Scaling of $\mathbb{R}^2$ by 3
- (Standard) shear of $\mathbb{R}^2$
- Orthogonal projection to the xy-plane in $\mathbb{R}^3$

Demo

Eigenvalues for rotations?

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)?

Demo

Summary of Section 5.1

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- Recipe: The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A
- Fact. If $v_1 \ldots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots \lambda_k$, then $\{v_1, \ldots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations

Typical exam questions 5.1

- Find the 2-eigenvectors for the matrix

$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

- True or false: The zero vector is an eigenvector for every matrix.
- How many different eigenvalues can there be for an $n \times n$ matrix?
- Consider the reflection of $\mathbb{R}^2$ about the line $y = 7x$. What are the eigenvalues (of the standard matrix)?
- Consider the $\pi/2$ rotation of $\mathbb{R}^3$ about the z-axis. What are the eigenvalues (of the standard matrix)?

- Confirm $\lambda = 1$ is not an eigenvalue of $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$

$A - 1 \cdot I = \begin{pmatrix} 1 & -4 \\ -1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ trivial Null space

So $\lambda = 1$ not an eigenvalue.