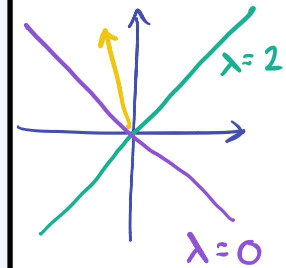
Eigenvectors and Eigenvalues

Confirming eigenvectors

2×2 matrix ⇒ at most 2 eigenvalues

So: We have found all eigenvals & eigenvectors for $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$



Poll

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ✓, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ✓, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ ✓, $\cancel{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$, $\cancel{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$ are eigenvectors of

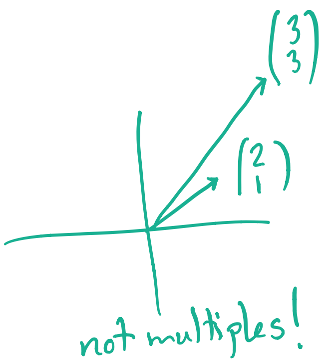
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ✓ \quad \lambda = 2$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \lambda = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{NO.}$$



Announcements Oct 26

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on Determinants II due Thursday night
- Quiz on Sections 4.2, 5.1 Friday 8 am - 8 pm EDT
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- My Office Hours Tue 11-12, Thu 9-10, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Chapter 5

Eigenvectors and eigenvalues

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or}$$
> $$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), fluid mixing, RLC circuits, clustering (data analysis), principal component analysis, Google, Netflix (collaborative prediction), infectious disease models, special relativity, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

If $v \neq 0$ &

In simpler terms: Av is a scalar multiple of v. then v is an eigenvector.

In other words: Av points in the same direction as v.

Think of this in terms of inputs and outputs!

eigen $=$ *characteristic (or: self)*

This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

Confirming eigenvalues

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

3 - eigenspace = Nul $(A - 3I)$

"

all eigenvectors

w/ eigenval 3

plus 0 vector.

$Av = 3v \quad v \neq 0$

$Av - 3Iv = 0 \quad v \neq 0$

$(A - 3I)v = 0 \quad v \neq 0$

$\rightsquigarrow$ Nul $(A - 3I)$

What is a general procedure for finding eigenvalues?

Eigenvectors and Eigenvalues

Confirming eigenvalues

So the recipe for checking if λ is an eigenvalue of A is:

- subtract λ from the diagonal entries of A $\longleftarrow$ $A - \lambda I$
- row reduce
- check if there are fewer than n pivots $\rightsquigarrow$ Nul $(A - \lambda I)$ nontrivial.

Confirm that $\lambda = 1$ is not an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

$$A - 1 \cdot I = \begin{pmatrix} 1 & -4 \\ -1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad 2 \text{ pivots.}$$

So no 1-eigenvectors.

Eigenspaces

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

Why is this a subspace?

Fact. λ-eigenspace for $A = \mathrm{Nul}(A - \lambda I)$

Example. Find the eigenspaces for $\lambda = 2$ and $\lambda = -1$ and sketch.

$$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

2-eigensp $= \mathrm{Nul}\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

-1-eigensp $= \mathrm{Nul}\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

Eigenspaces

Bases

Find a $\boxed{\text{basis for the 2–eigenspace}}$ = basis for Nul $A-2I$

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

$$A-2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rightsquigarrow$ vect param form

2 vectors.

$\rightsquigarrow$ 2D eigenspace.

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x = x$
$y = y$
$z = 0$

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2-eigenspace is xy-plane.

(Other eigenvalue: 0   0-eigenspace is z-axis.)

$$A \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Q. Find the 2-eigenspace: $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

A. $A - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightsquigarrow$ 2-eigenspace is all of $\mathbb{R}^2$

Eigenvalues

And invertibility

Fact. A invertible $\Leftrightarrow 0$ is not an eigenvalue of A

Why?

0 is an eigenvalue

$\Longleftrightarrow Av = 0 \cdot v = 0 \quad v \neq 0,$

$\Longleftrightarrow$ not invertible

example: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

$$A = \begin{pmatrix} 3 & 5 \\ 0 & 7 \end{pmatrix}$$

$$A - 3I = \begin{pmatrix} 0 & 5 \\ 0 & 4 \end{pmatrix} \qquad 1 \text{ pivot}$$

$$A - 7I = \begin{pmatrix} -4 & 5 \\ 0 & 0 \end{pmatrix} \qquad 1 \text{ pivot.}$$

Important! You can not find the eigenvalues by row reducing first! After you find the eigenvalues, you row reduce $A - \lambda I$ to find the eigenspaces. But once you start row reducing the original matrix, you change the eigenvalues.

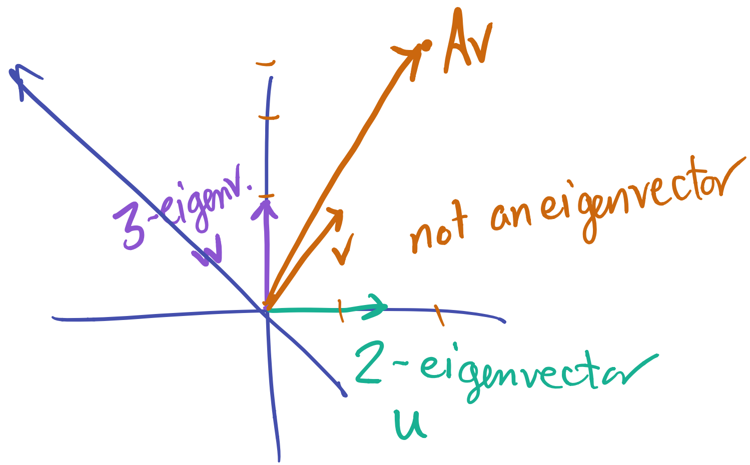
$$\begin{pmatrix} 3 & 5 \\ 0 & 7 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{different eigenvalues!}$$

Eigenvalues

Distinct eigenvalues

Fact. If $v_1 \dots v_k$ are ~~distinct~~ eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.

Why?



$$v = u + w$$

$$\begin{aligned} Av &= A(u+w) \\ &= Au + Aw \\ &= 2u + 3w \end{aligned}$$

Consequence: At most n distinct eigenvalues for $n \times n$ matrix.

Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

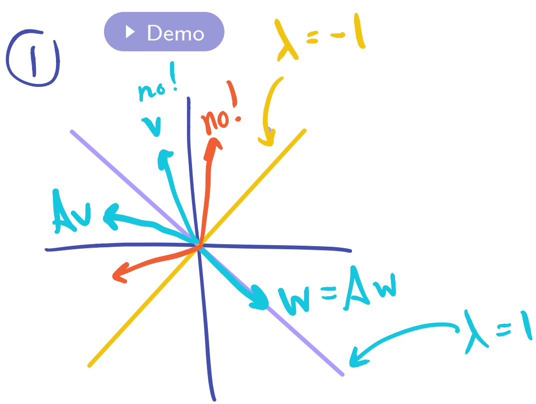
Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

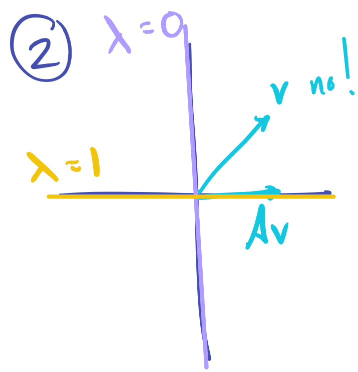
1. Reflection about the line $y = -x$ in $\mathbb{R}^2$
2. Orthogonal projection onto the x-axis in $\mathbb{R}^2$
3. Scaling of $\mathbb{R}^2$ by 3 like $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
4. (Standard) shear of $\mathbb{R}^2$
5. Orthogonal projection to the xy-plane in $\mathbb{R}^3$

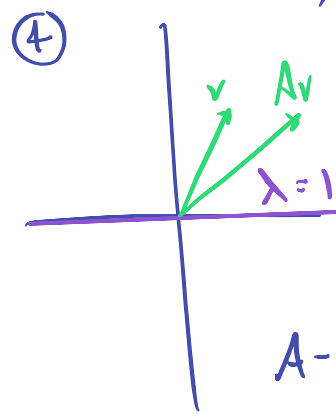
③ $\lambda = 3$
3-eigensp $= \mathbb{R}^2$

⑤ $\lambda = 1 \rightsquigarrow$ xy-plane $=$ 1-eigensp

$\lambda = 0 \rightsquigarrow$ z-axis $=$ 0-eigensp







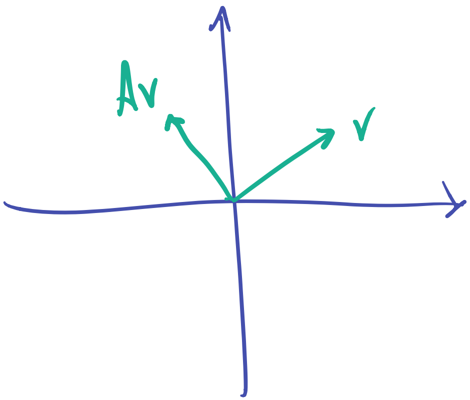
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ eigenvals: 1 (triang matrix)

$A - 1 \cdot I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Eigenvalues for rotations?

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)?



no eigenvalues or eigenvectors.

(actually there is a complex eigenvalue).

Demo

Summary of Section 5.1

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- Recipe: The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A
- Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations

Typical exam questions 5.1

- Find the 2-eigenvectors for the matrix

$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

- True or false: The zero vector is an eigenvector for every matrix.
- How many different eigenvalues can there be for an $n \times n$ matrix?
- Consider the reflection of $\mathbb{R}^2$ about the line $y = 7x$. What are the eigenvalues (of the standard matrix)?
- Consider the $\pi/2$ rotation of $\mathbb{R}^3$ about the z-axis. What are the eigenvalues (of the standard matrix)?