Announcements Oct 28

- Please turn on your camera if you are able and comfortable doing so
- WeBWorK on Determinants II due Thursday night
- Quiz on Sections 4.2, 5.1 Friday 8 am - 8 pm EDT
- Third Midterm Friday Nov 20 8 am - 8 pm on §4.1-5.6
- My Office Hours Tue 11-12, Thu 9-10, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`

Chapter 5

Eigenvectors and eigenvalues

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or}$$
> $$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), fluid mixing, RLC circuits, clustering (data analysis), principal component analysis, Google, Netflix (collaborative prediction), infectious disease models, special relativity, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

In simpler terms: Av is a scalar multiple of v.

In other words: Av points in the same direction as v.

Think of this in terms of inputs and outputs!

eigen $=$ *characteristic (or: self)*

This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

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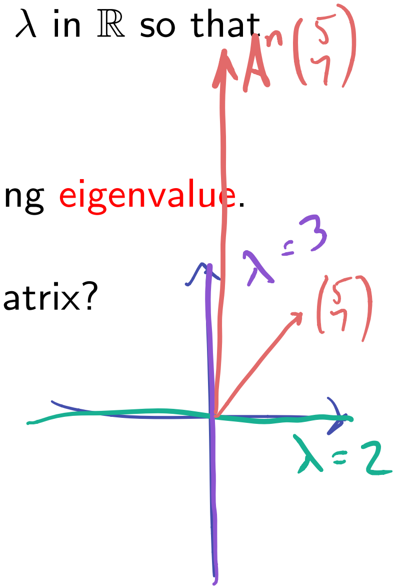


Can you find any eigenvectors/eigenvalues for the following matrix?

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^n \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 2^n \cdot 5 \\ 3^n \cdot 7 \end{pmatrix}$$

What happens when you apply larger and larger powers of A to a vector?

slope: $\dfrac{3^n \cdot 7}{2^n \cdot 5} \longrightarrow \infty$

A pulling towards y-axis

Also A multiplies vectors by 3 every time (n big)

Rabbits

What's up with them?

Eigenvectors and Eigenvalues

When we apply large powers of the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

to a vector v not on the x-axis, we see that $A^n v$ gets closer and closer to the y-axis, and it's length gets approximately tripled each time. This is because the largest eigenvalue is 3 and its eigenspace is the y-axis.

For the rabbit matrix

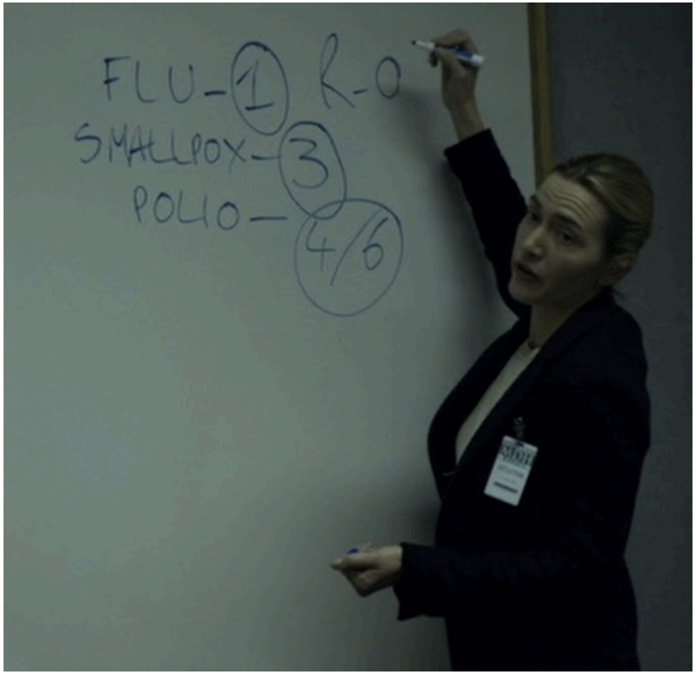
$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

We will see that 2 is the largest eigenvalue, and its eigenspace is the span of the vector $(16, 4, 1)$. That's why all populations of rabbits tend towards the ratio 16:4:1 and why the population approximately doubles each year.

R_0

R_0

For a given virus, R_0 is the average number of people that each infected person infects. If R_0 is large, that is bad. Patient zero infects R_0 people, who then infect R_0^2 people, who then infect R_0^3 people. That is exponential growth. (If R_0 is less than 1, then that's good.)



R_0

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Whenever we see an exponential growth rate, we should think: eigenvalue.

It turns out that R_0 is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment. That's a matrix. The largest eigenvalue is R_0.

R_0 is an eigenvalue

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For malaria, the compartments might be mosquitoes and humans.

For a sexually transmitted disease in a heterosexual population, the compartments might be males and females.

R_0 is an eigenvalue

It turns out that R_0 is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment.

The SIR model has compartments for Susceptible, Infected, and Recovered.

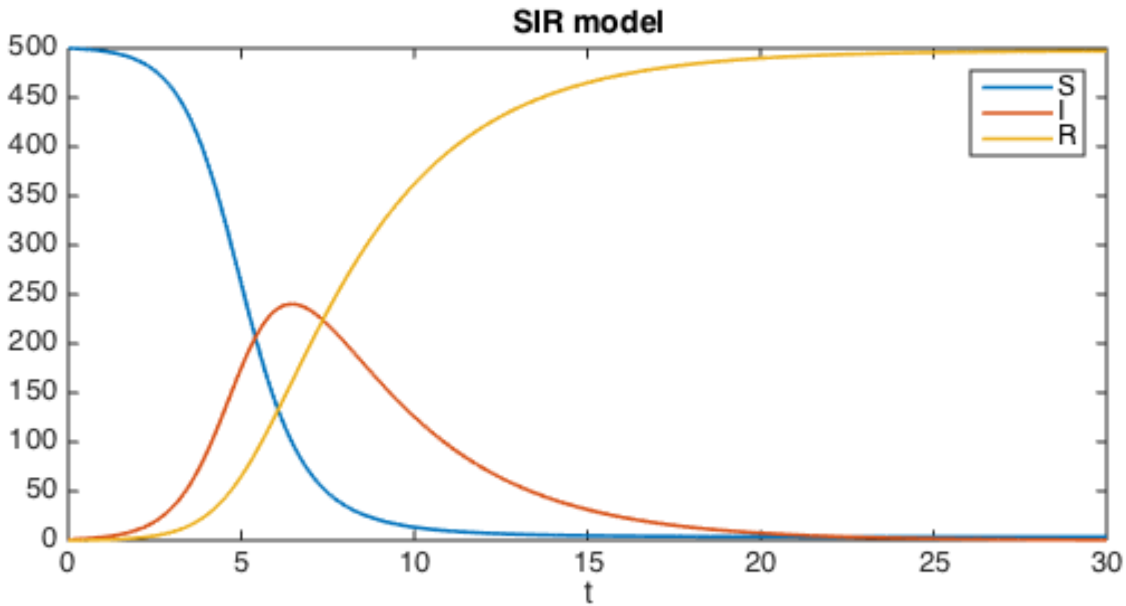
The arrows are governed by differential equations (Math 2552). Why do the labels on the arrows make sense? (The greek letters are constants).

There is a nice discussion of this by James Holland Jones (Stanford).

▸ Paper

Bell curves

The growth rate of infection does not stay exponential forever, because the recovered population has immunity. That's where you get these bell curves.



Section 5.2

The characteristic polynomial

Outline of Section 5.2

- How to find the eigenvalues, via the characteristic polynomial
- Techniques for the 3×3 case

Characteristic polynomial

Recall:

$$\lambda \text{ is an eigenvalue of } A \iff A - \lambda I \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial, the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

The eigenrecipe

Say you are given an square matrix A.

Step 1. Find the eigenvalues of A by solving

$$\det(A - \lambda I) = 0$$

Step 2. For each eigenvalue λ_i the λ_i-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

To find a basis, find the vector parametric solution, as usual.

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

<u>Eigenvalues</u>

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0$$

$$(5-\lambda)(1-\lambda) - 4 = 0$$

$$\lambda^2 - 6\lambda + 1 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$$

<u>Eigenspaces</u>

$$\lambda = 3 + 2\sqrt{2}$$

$$A - \lambda I = \begin{pmatrix} 5-(3+2\sqrt{2}) & 2 \\ 2 & 1-(3+2\sqrt{2}) \end{pmatrix}$$

$$\xrightarrow[\text{red}]{\text{row}} \begin{pmatrix} 2-2\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix}$$

I know the bottom row is a mult. of top!

$$\rightsquigarrow \begin{pmatrix} -2 \\ 2-\sqrt{2} \end{pmatrix}$$

Two shortcuts for 2×2 eigenvectors

Find the eigenspaces for the eigenvalues on the last page. Two tricks.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

3×3 matrices

The 3×3 case is harder. There is a version of the quadratic formula for cubic polynomials, called Cardano's formula. But it is more complicated. It looks something like this:

$$\begin{aligned} x &= \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ &+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}. \end{aligned}$$

There is an even more complicated formula for quartic polynomials.

One of the most celebrated theorems in math, the Abel–Ruffini theorem, says that there is no such formula for quintic polynomials.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}$$

prev slide says:

$-\lambda^3 + 8\lambda^2 + ??\lambda + \det$

What are the eigenvalues? $\boxed{\text{Hint: Don't multiply everything out!}}$

$$\det \begin{pmatrix} 7-\lambda & 0 & 3 \\ -3 & 2-\lambda & -3 \\ -3 & 0 & -1-\lambda \end{pmatrix} = (2-\lambda)\big((7-\lambda)(-1-\lambda)+9\big)$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

2 is an eigenvalue!

$$= (2-\lambda)(\lambda^2 - 6\lambda + 2)$$

$\lambda = 2$, $\lambda = \dfrac{6 \pm \sqrt{28}}{2}$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda + 0$

What are the eigenvalues?

$-\lambda(\lambda^2 - 9\lambda + 8)$

$-\lambda(\lambda - 8)(\lambda - 1)$

$\lambda = 0, 8, 1$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\boxed{-\lambda^2}$$

$$\lambda - 2 \,\overline{\big)\, -\lambda^3 \qquad +3\lambda + 2}$$

$$-\quad -\lambda^3 + 2\lambda^2$$

$$-2\lambda^2 + 3\lambda + 2$$

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

We know from experiments $\lambda = 2$ is eigenvalue

So $(\lambda - 2)$ is a factor.

Answer:

etc.

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

$$(\lambda - 2)(-\lambda^2 + \_\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Without the hint, could use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial and eigenvalues.

$$\begin{pmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{pmatrix}$$

Characteristic polynomial: $-\lambda^3 + 9\lambda^2 - 23\lambda + 15$

This time we don't know any of the roots! We can use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

So we plug in ± 1, ± 3, ± 5, ± 15 into the polynomial and hope for the best. Luckily we find that 1, 3, and 5 are all roots, so we found all the eigenvalues!

If we were less lucky and found only one eigenvalue, we could again use long division like on the last slide.

Eigenvalues

Triangular matrices

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

Char poly:

$$(1-\lambda)(4-\lambda)(6-\lambda)$$

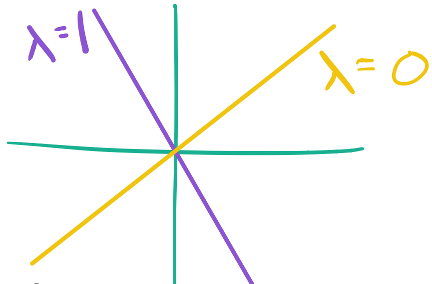
$$\det \begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{pmatrix}$$

Warning! You cannot find eigenvalues by row reducing and then using this fact. You need to work with the original matrix. Finding eigenspaces involves row reducing $A - \lambda I$, but there is no row reduction in finding eigenvalues.

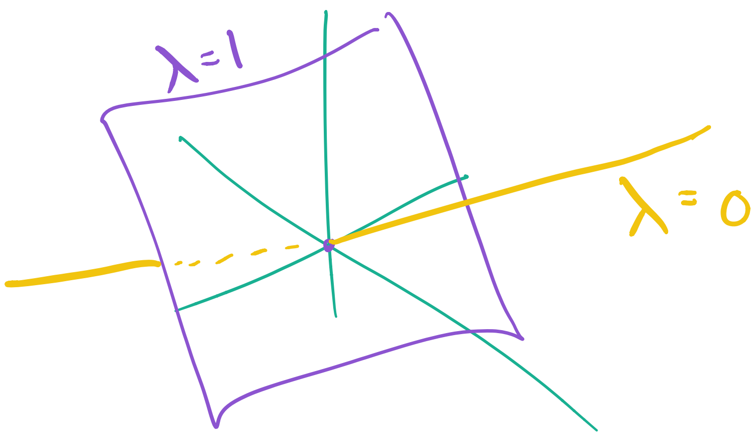
Eigenvalues

Geometrically defined matrices

Say that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that projects onto the plane $2x + 3y = 0$ and that A is the standard matrix for T. What are the eigenvalues of A?



Say that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation that projects onto the plane $2x + 3y - z = 0$ and that A is the standard matrix for T. What are the eigenvalues of A?



Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

trace $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 + 5 + 9 = 15$

The characteristic polynomial always looks like:

$$(-1)^n \lambda^n + (-1)^{n-1} \boxed{\text{trace(A)}} \lambda^{n-1} + \boxed{???} \lambda^{n-2} + \cdots \boxed{???} \lambda + \boxed{\det(A)}$$

$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$

$\lambda^2 - \underbrace{6}_{\text{trace}}\lambda + \underbrace{1}_{\det}$

So for a 2×2 matrix:

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A)$$

And for a 3×3 matrix:

$$-\lambda^3 + \mathrm{trace}(A)\lambda^2 - \boxed{???} \lambda + \det(A)$$

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{0} & 0 & 0 \\ 0 & 0 & \boxed{-1} & 0 \\ 0 & 0 & 0 & \boxed{0} \end{pmatrix}$$

Char poly

$(1-\lambda)^1 \lambda^2 (-1-\lambda)^1$

0 is a root twice, so it has alg. multiplicity 2.

eigenvalues: 1, 0, −1

−1, 1 has alg mult 1
0 has alg mult 2.

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

Why? Degree of char poly is n.

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor ✓
  - If there is no constant term then factor out λ ✓
  - If the matrix is triangular, the eigenvalues are the diagonal entries ✓
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

rat'l root thm.

$-\lambda^3 + 3\lambda + 2$ guess roots $\pm 1, \pm 2$

(divisors of const. term)

Plug in, discover 2 is a root

Divide by $(\lambda - 2)$ to find other roots.

Typical Exam Questions 5.2

- True or false: Every $n \times n$ matrix has an eigenvalue.
- True or false: Every $n \times n$ matrix has n distinct eigenvalues.
- True or false: The nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the $n \times n$ zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

$(\lambda+2)(\lambda+i)(\lambda-i)$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues.