Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 10, 20)$, $(0, -1, -2)$
2. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 1, 0)$, $(0, 0, \sqrt{2})$
3. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 5, 7)$, $(0, 6, 8)$
4. $(0, 1, 2)$ is in the span of $(5, 7, 0)$, $(6, 8, 0)$, $(3, 3, 4)$

1. $(0,1,2) = 0\cdot(3,3,4) + \frac{1}{10}\cdot(0,10,20) + 0\cdot(0,-1,-2)$
2. $(0,1,2) = 1\cdot(0,1,0) + \sqrt{2}\cdot(0,0,\sqrt{2})$
3. $(0,5,7)$ & $(0,6,8)$ span yz-plane & $(0,1,2)$ in yz-plane
4. Span of $(5,7,0)$ & $(6,8,0)$ is xy-plane.
   And $(3,3,4)$ not in xy-plane, so span of all 3 is $\mathbb{R}^3$

Announcements Sep 9

- WeBWorK on Sections 1.2 and 1.3 due Thursday night
- Quiz on Sections 1.2 and 1.3 Friday 8 am - 8 pm EDT
- First Midterm Sep 18
- My office hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Juntao Thu 3-4
- Studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Some typical comments from the survey

- The difficulty of the homework. It seemed way more challenging than anything we did in class and on the quiz. I would prefer more homework overall, but easier questions.
- I think the studios don't really help me as much as I hoped they would, mostly because many people in the studio do not participate in the breakout session when we are split into small groups. I kind of wish it was like a class session but more based on the TA going over the worksheet with us. I feel like with the breakout sessions, we end up not having enough time to go over the whole worksheet.
- Answering questions in-between lecture notes can get confusing. Also, the pace is a bit fast at times...
- Teams keeps crashing on my laptop so I am often unable to access the class material.

Section 2.4

Solution Sets

\# pivots of A

= dim. of span of cols of A.

If $Ax = b$ consistent,

\# cols of A w/o pivots

= dim of soln set

= # free vars

= #vectors in param form

Outline

- Understand the geometric relationship between the solutions to $Ax = b$ and $Ax = 0$
- Understand the relationship between solutions to $Ax = b$ and spans
- Learn the parametric vector form for solutions to $Ax = b$

Homogeneous systems

Solving $Ax = b$ is easiest when $b = 0$. Such equations are called homogeneous.

Homogenous systems are always consistent. *Why?*

x = 0 is a solution.

When does $Ax = 0$ have a nonzero/nontrivial solution?

A has a column w/o pivot.

If there are k-free variables and n total variables, then the solution is a k-dimensional plane through the origin in $\mathbb{R}^n$. In particular it is a span.

Same

Parametric Vector Forms for Solutions

Homogeneous case

can omit last column:

Solve the matrix equation $Ax = 0$ where

$$A = \left(\begin{array}{rrrr|r} 1 & 2 & 0 & -1 & 0 \\ -2 & -3 & 4 & 5 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{array}\right) \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & -8 & -7 \\ 0 & \boxed{1} & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We already know the parametric form:

$$\begin{aligned} x_1 &= 8x_3 + 7x_4 \\ x_2 &= -4x_3 - 3x_4 \\ x_3 &= x_3 \quad \text{(free)} \\ x_4 &= x_4 \quad \text{(free)} \end{aligned}$$

We can also write this in parametric vector form:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

\# vectors here
= \# free vars
= dim. of soln set

Or we can write the solution as a span: $\mathrm{Span}\{(8, -4, 1, 0), (7, -3, 0, 1)\}$.

Parametric Vector Forms for Solutions

Homogeneous case

Find the parametric vector form of the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$ already reduced

param. form

$$\left.\begin{aligned} x_1 &= -x_2 - x_3 - x_4 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}\right\} \text{free}$$

param. vector form

$$x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

As a span:

$$\mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \text{3D-plane thru } 0 \text{ in } \mathbb{R}^4$$

Variables, equations, and dimension

Poll

For $b \neq 0$, the solutions to $Ax = b$ are...

1. always a span
2. sometimes a span
3. never a span

Nonhomogeneous Systems

Suppose $Ax = b$ and $b \neq 0$.

As before, we can find the parametric vector form for the solution in terms of free variables.

What is the difference?

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form of the solution to $Ax = b$ where:

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 3 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -8 & -7 & -13 \\ 0 & 1 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We already know the parametric form:

$$\begin{aligned} x_1 &= -13 + 8x_3 + 7x_4 \\ x_2 &= 8 - 4x_3 - 3x_4 \\ x_3 &= x_3 \quad \text{(free)} \\ x_4 &= x_4 \quad \text{(free)} \end{aligned}$$

We can also write this in parametric vector form:

Parallel translates the plane

plane through origin

$$\begin{pmatrix} -13 \\ 8 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

2D plane in $\mathbb{R}^4$ not through 0.

This is a translate of a span: $(-13, 8, 0, 0) + \mathrm{Span}\{(8, -4, 1, 0), (7, -3, 0, 1)\}$.

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector form for the solution to $Ax = (9)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 9 \end{array}\right)$$

param. form

$$\begin{aligned} x_1 &= 9 - x_2 - x_3 - x_4 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

free

param. vector form

$$\begin{pmatrix} 9 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

As a span:

$$\begin{pmatrix} 9 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$ = 3D-plane not thru 0 in $\mathbb{R}^4$

Homogeneous vs. Nonhomogeneous Systems

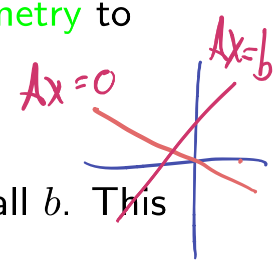
Key realization. Set of solutions to $Ax = b$ obtained by taking one solution and adding all possible solutions to $Ax = 0$.

$$Ax = 0 \text{ solutions} \rightsquigarrow Ax = b \text{ solutions}$$

$$x_k v_k + \cdots + x_n v_n \rightsquigarrow p + x_k v_k + \cdots + x_n v_n$$

$$\begin{pmatrix} 9 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

So: set of solutions to $Ax = b$ is parallel to the set of solutions to $Ax = 0$. It is a translate of a plane through the origin. (Again, we are using geometry to understand algebra!)



So by understanding $Ax = 0$ we gain understanding of $Ax = b$ for all b. This gives structure to the set of equations $Ax = b$ for all b.

Example. A = some 2×2 matrix. If solns to $Ax=0$ is $x+2y=0$ Can I find a b so solns to $Ax=b$ is $x+y=1$? No. Not parallel.

Demo

Demo

Two different things

Suppose A is an $m \times n$ matrix. Notice that if $Ax = b$ is a matrix equation then x is in $\mathbb{R}^n$ and b is in $\mathbb{R}^m$. There are two different problems to solve.

dim = # non-pivots

1. If we are given a specific b, then we can solve $Ax = b$. This means we find all x in $\mathbb{R}^n$ so that $Ax = b$. We do this by row reducing, taking free variables for the columns without pivots, and writing the (parametric) vector form for the solution.

The collection of all good b's is a plane thru 0, dim = # pivots

2. We can also ask for which b in $\mathbb{R}^m$ does $Ax = b$ have a solution? The answer is: when b is in the span of the columns of A. So the answer is "all b in $\mathbb{R}^m$" exactly when the span of the columns is $\mathbb{R}^m$ which is exactly when A has m pivots.

If you go back to the ▸ Demo from earlier in this section, the first question is happening on the left and the second question on the right.

Example. Say that $A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$. We can ask: (1) Does $Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ have a solution? and (2) ~~What are all the solutions?~~

For which b does $Ax = b$ have a soln? → when b in span of cols of A

Summary of Section 2.4

- The solutions to $Ax = 0$ form a plane through the origin (span)
- The solutions to $Ax = b$ form a plane not through the origin
- The set of solutions to $Ax = b$ is parallel to the one for $Ax = 0$
- In either case we can write the parametric vector form. The parametric vector form for the solution to $Ax = 0$ is obtained from the one for $Ax = b$ by deleting the constant vector. And conversely the parametric vector form for $Ax = b$ is obtained from the one for $Ax = 0$ by adding a constant vector. This vector translates the solution set.

Typical exam questions

- Suppose that the set of solutions to $Ax = b$ is the plane $z = 1$ in $\mathbb{R}^3$. What is the set of solutions to $Ax = 0$?
- Suppose that the set of solutions to $Ax = 0$ is the line $y = x$ in $\mathbb{R}^2$. Is it possible that there is a b so that the set of solutions to $Ax = b$ is the line $x + y = 1$?
- Suppose that the set of solutions to $Ax = b$ is the plane $x + y = 1$ in $\mathbb{R}^3$. Is is possible that there is a b so that the set of solutions to $Ax = b$ is the z-axis?
- Suppose that the set of solutions to $Ax = 0$ is the plane $x + 2y - 3z = 6$ in $\mathbb{R}^3$ and that the vector $(1, 3, 5)$ is a solution to $Ax = b$. Find one other solution to $Ax = b$. Find all of them.
- Is there a 2×2 matrix so that the set of solutions to $Ax = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ is the line $y = x + 1$? If so, find such an A. If not, explain why not.

Section 2.5

Linear Independence

Section 2.5 Outline

- Understand what is means for a set of vectors to be linearly independent
- Understand how to check if a set of vectors is linearly independent

Linear Independence

$\mathrm{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}3\\6\end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$

Basic question: What is the smallest number of vectors needed in the parametric solution to a linear system? A lin. ind set of vectors.

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

homogeneous $\rightsquigarrow$ has 0 soln.

has only the trivial solution. It is linearly dependent otherwise.

Indep. Example $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ $\quad \underline{\text{has to be } 0}\begin{pmatrix}1\\0\end{pmatrix} + \underline{\text{has to be } 0}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

So, linearly dependent means there are $x_1, x_2, \ldots, x_k$ not all zero so that

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

This is a *linear dependence* relation.

Dep. Example $\left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}3\\6\end{pmatrix}\right\}$ because $\underline{-3}\begin{pmatrix}1\\2\end{pmatrix} + \underline{1}\begin{pmatrix}3\\6\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Linear Independence

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

vector eqn.

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & & \uparrow \\ v_1 & v_2 & v_3 & \cdots & v_k \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{pmatrix} x = 0$$

has only the trivial solution.

Fact. The columns of A are linearly independent
$\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
$\Leftrightarrow$ A has a pivot in each column

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ lin ind?

defn. of lin ind.

Why?

does

$x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

have only 0 soln

vect eqn

$\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

matrix eqn

has only 0 soln?

from vec eqn to matrix eqn

row reduce
get col w/o pivot
So non-0 soln
so dep.

Linear Independence

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Linear Independence

When is $\{v\}$ is linearly dependent?

When is $\{v_1, v_2\}$ is linearly dependent?

When is the set $\{v_1, v_2, \ldots, v_k\}$ linearly dependent?

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if they span a k-dimensional plane. (algebra $\leftrightarrow$ geometry)

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if we can remove a vector from the set without changing the dimension of the span.

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some v_i lies in the span of $v_1, \ldots, v_{i-1}$.

Demo

Span and Linear Independence

Is $\left\{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Try using the last fact: the set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some v_i lies in the span of $v_1, \ldots, v_{i-1}$.

Linear independence and free variables

Theorem. Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and consider the vector equation

$$x_1 v_1 + \cdots + x_k v_k = 0.$$

The set of vectors corresponding to non-free variables are linearly independent.

So, given a bunch of vectors $v_1, \ldots, v_k$, if you want to find a collection of v_i that are linearly independent, you put them in the columns of a matrix, row reduce, find the pivots, and then take the original v_i corresponding to those columns.

Example. Try this with $(1, 1, 1)$, $(2, 2, 2)$, and $(1, 2, 3)$.

Linear independence and coordinates

Fact. If $v_1, \ldots, v_k$ are linearly independent vectors then we can write each element of

$$\mathrm{Span}\{v_1, \ldots, v_k\}$$

in exactly one way as a linear combination of $v_1, \ldots, v_k$.

More on this later, when we get to bases.

Fact 1. Say $v_1, \ldots, v_k$ are in $\mathbb{R}^n$. If $k > n$, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

Fact 2. If one of $v_1, \ldots, v_k$ is 0, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

Parametric vector form and linear independence

Poll

Say you find the parametric vector form for a homogeneous system of linear equations, and you find that the set of solutions is the span of certain vectors. Then those vectors are...

1. always linearly independent
2. sometimes linearly independent
3. never linearly independent

Example. In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Parametric Vector Forms and Linear Independence

In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors that appear are linearly independent (why?). This means that we can't write the solution with fewer than two vectors (why?). This also means that this way of writing the solution set is efficient: for each solution, there is only one choice of x_3 and x_4 that gives that solution.

Summary of Section 2.5

- A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

has only the trivial solution. It is linearly dependent otherwise.

- The cols of A are linearly independent
  $\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
  $\Leftrightarrow$ A has a pivot in each column
- The number of pivots of A equals the dimension of the span of the columns of A
- The set $\{v_1, \ldots, v_k\}$ is linearly independent $\Leftrightarrow$ they span a k-dimensional plane
- The set $\{v_1, \ldots, v_k\}$ is linearly dependent $\Leftrightarrow$ some v_i lies in the span of $v_1, \ldots, v_{i-1}$.
- To find a collection of linearly independent vectors among the $\{v_1, \ldots, v_k\}$, row reduce and take the (original) v_i corresponding to pivots.

Typical exam questions

- State the definition of linear independence.
- *Always/sometimes/never.* A collection of 99 vectors in $\mathbb{R}^{100}$ is linearly dependent.
- *Always/sometimes/never.* A collection of 100 vectors in $\mathbb{R}^{99}$ is linearly dependent.
- Find all values of h so that the following vectors are linearly independent:

$$\left\{ \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ h \end{pmatrix} \right\}$$

- *True/false.* If A has a pivot in each column, then the rows of A are linearly independent.
- *True/false.* If u and v are vectors in $\mathbb{R}^5$ then $\{u, v, \sqrt{2}u - \pi v\}$ is linearly independent.
- If you have a set of linearly independent vectors, and their span is a line, how many vectors are in the set?