Variables, equations, and dimension

Poll

For $b \neq 0$, the solutions to $Ax = b$ are...

1. always a span
2. sometimes a span
3. never a span

Span = plane thru 0.

If 0 is a soln to $Ax = b$

then $A \cdot 0 = b$

$0 = b$.

Announcements Sep 14

See Resources on Canvas Front page.

- WeBWorK on Sections 2.4 and 2.5 due Thursday night
- First Midterm Friday 8 am - 8 pm Sec 1.1 - 2.5
- My office hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Juntao Thu 3-4

Practice Exam on Canvas Wed.

- Review session(s) tba
- ~~Studio on Friday will be an office hour.~~ No Studio Fri
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Chapter 2

System of Linear Equations: Geometry

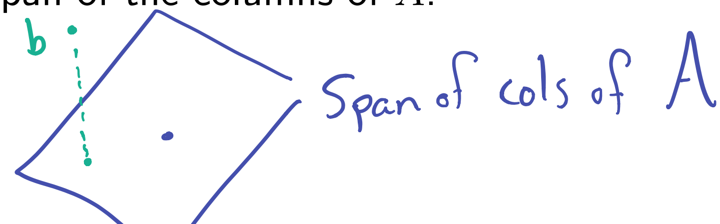
Where are we?

In Chapter 1 we learned how to completely solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution.

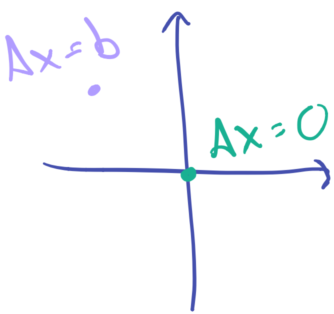
In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning. For instance...

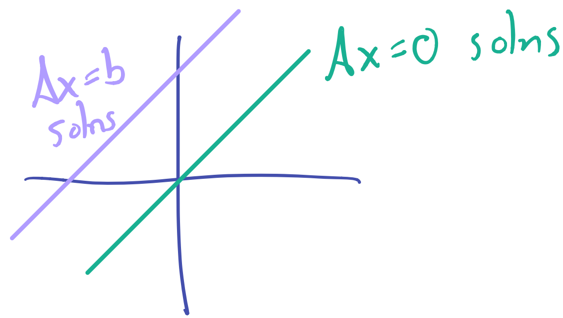
$Ax = b$ is consistent $\Leftrightarrow$ b is in the span of the columns of A.

inconsistent:



The solutions to $Ax = b$ are parallel to the solutions to $Ax = 0$.





Section 2.4

Solution Sets

Parametric Vector Forms for Solutions

Nonhomogeneous case

Find the parametric vector forms for $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

...and $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

Section 2.5

Linear Independence

Section 2.5 Outline

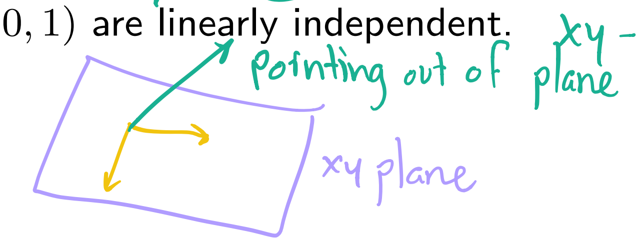
- Understand what is means for a set of vectors to be linearly independent
- Understand how to check if a set of vectors is linearly independent

Linear Independence

The idea of linear independence: a collection of vectors $v_1, \ldots, v_k$ is linearly independent if they are all pointing in truly different directions. This means that none of the v_i is in the span of the others.

dim of span = # vectors

For example, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are linearly independent.



sum

Also, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$ are linearly dependent.

$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}2\\3\\4\end{pmatrix}$

dependent

What is this good for? A basic question we can ask about solving linear equations is: What is the smallest number of vectors needed in the parametric solution to a linear system? We need linear independence to answer this question. See the last slide in this section.

Linear Independence

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

0 soln

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

has only the trivial solution. It is linearly dependent otherwise.

dep

$$\boxed{0\begin{pmatrix}1\\0\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}} \qquad \boxed{1\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\0\end{pmatrix} + (-1)\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}}$$

indep

So, linearly dependent means there are $x_1, x_2, \ldots, x_k$ not all zero so that

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

This is a *linear dependence* relation.

$$-2\begin{pmatrix}1\\0\\0\end{pmatrix} + 1\begin{pmatrix}2\\0\\0\end{pmatrix} + 0\begin{pmatrix}15\\19\\-8\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

solns: $(1,1,-1)$

$(5,5,-5)$

$\vdots$

$(-z,-z,z)$

nontrivial soln $\Longleftrightarrow$ dependent

Linear Independence

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

has only the trivial solution.

Fact. The columns of A are linearly independent
$\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
$\Leftrightarrow$ A has a pivot in each column
$\Leftrightarrow$ no free vars

Why?

$$Ax = 0$$

So: Given some $v_1, \ldots, v_k$
If you want to know if they are indep, make a matrix, row reduce.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$$

nontriv. soln, dep cols

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x = 0$$

only triv. soln. indep.

Linear Independence

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row red}} \begin{pmatrix} \boxed{1} & \cancel{*} & \cancel{*} \\ 0 & \boxed{1} & \cancel{*} \\ 0 & 0 & 0 \end{pmatrix}$$

no pivot

$$2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

Depend.

or $$2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

h

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{} & & \\ & \boxed{} & \\ & & \boxed{} \end{pmatrix}$$

indep.

Linear Independence

When is $\{v\}$ is linearly dependent?

$V = 0$ -vector

$5 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$0 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ indep.

When is $\{v_1, v_2\}$ is linearly dependent?

multiples/ collinear.

When is the set $\{v_1, v_2, \ldots, v_k\}$ linearly dependent?

the answer is not: if they multiples/collinear

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if they span a k-dimensional plane. (algebra $\leftrightarrow$ geometry)

Almost same

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if we can remove a vector from the set without changing (the dimension of) the span.

example. $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

Almost same

Fact. The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some v_i lies in the span of $v_1, \ldots, v_{i-1}$.

So check Is v_2 in $\mathrm{Span}\{v_1\}$

Is v_3 in $\mathrm{Span}\{v_1, v_2\}$

Is v_4 in $\mathrm{Span}\{v_1, v_2, v_3\}$

Demo

Span and Linear Independence

Is $\left\{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Could also row reduce, find 3 pivots.

Try using the last fact: the set $\{v_1, v_2, \dots, v_k\}$ is linearly dependent if and only if some v_i lies in the span of $v_1, \dots, v_{i-1}$.

$\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}$ is nonzero ✓

$\begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}$ not in span$\left\{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix} \right\}$ ✓ (not multiples)

$\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ not in span$\left\{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix} \right\}$ ✓ because of the 4

So Independent!

Linear independence and free variables

Theorem. Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and consider the vector equation

$$x_1 v_1 + \cdots + x_k v_k = 0.$$

The set of vectors corresponding to non-free variables are linearly independent.

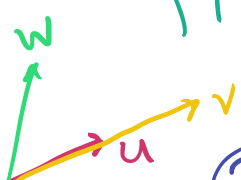
So, given a bunch of vectors $v_1, \ldots, v_k$, if you want to find a collection of v_i that are linearly independent, you put them in the columns of a matrix, row reduce, find the pivots, and then take the original v_i corresponding to those columns.

Example. Try this with $(1, 1, 1)$, $(2, 2, 2)$, and $(1, 2, 3)$.

dep. or indep?

To make a indep set, delete 2^{nd} vector.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} \square & * & * \\ 0 & 0 & \square \\ 0 & 0 & 0 \end{pmatrix}$$



$2u - v = 0$

$v = 2u$

$u = v/2$

So first & third vectors are indep, second vector dep. on them.

Linear independence and coordinates

Fact. If $v_1, \ldots, v_k$ are linearly independent vectors then we can write each element of

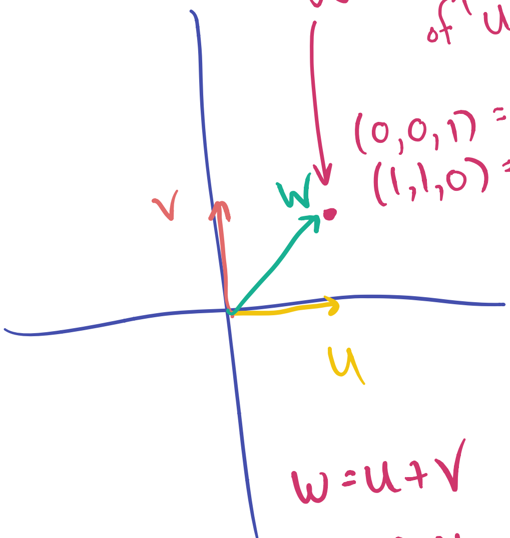
$$\mathrm{Span}\{v_1, \ldots, v_k\}$$

in exactly one way as a linear combination of $v_1, \ldots, v_k$.

∞ many "addresses" in terms of u, v, w

$(0,0,1) = 0 \cdot u + 0 \cdot v + 1 \cdot w$

$(1,1,0) = 1 \cdot u + 1 \cdot v + 0 \cdot w$

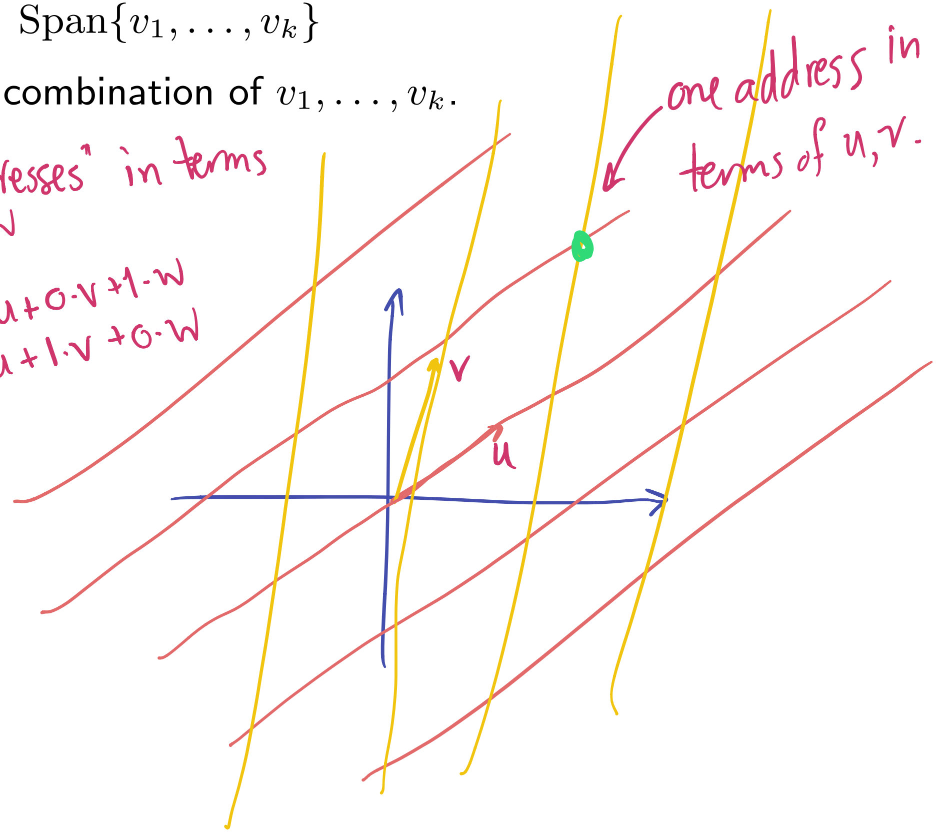


$w = u + v$

$v = w - u$

$u = w - v$

one address in terms of u, v.



Span and Linear Independence

Two More Facts

Fact 1. Say $v_1, \ldots, v_k$ are in $\mathbb{R}^n$. If $\boxed{k > n}$, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

e.g. 3 vectors in $\mathbb{R}^2$ $\begin{pmatrix} \square & & \\ & \square & \end{pmatrix}$

no pivot.

Fact 2. If one of $v_1, \ldots, v_k$ is 0, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

Parametric vector form and linear independence

Poll

Say you find the parametric vector form for a homogeneous system of linear equations, and you find that the set of solutions is the span of certain vectors. Then those vectors are...

1. always linearly independent
2. sometimes linearly independent
3. never linearly independent

Example. In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Parametric Vector Forms and Linear Independence

In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors that appear are linearly independent (why?). This means that we can't write the solution with fewer than two vectors (why?). This also means that this way of writing the solution set is efficient: for each solution, there is only one choice of x_3 and x_4 that gives that solution.

$A x = \underline{b}$

✓ dim of solns = # cols w/o pivot

✓ dim of span of cols = # cols w/ pivot

Summary of Section 2.5

- A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

  has only the trivial solution. It is linearly dependent otherwise.
- The cols of A are linearly independent
  $\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
  $\Leftrightarrow$ A has a pivot in each column
- The number of pivots of A equals the dimension of the span of the columns of A
- The set $\{v_1, \ldots, v_k\}$ is linearly independent $\Leftrightarrow$ they span a k-dimensional plane
- The set $\{v_1, \ldots, v_k\}$ is linearly dependent $\Leftrightarrow$ some v_i lies in the span of $v_1, \ldots, v_{i-1}$.
- To find a collection of linearly independent vectors among the $\{v_1, \ldots, v_k\}$, row reduce and take the (original) v_i corresponding to pivots.

Typical exam questions

- State the definition of linear independence.
- *Always/sometimes/never.* A collection of 99 vectors in $\mathbb{R}^{100}$ is linearly dependent.
- *Always/sometimes/never.* A collection of 100 vectors in $\mathbb{R}^{99}$ is linearly dependent.
- Find all values of h so that the following vectors are linearly independent:

$$\left\{ \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ h \end{pmatrix} \right\}$$

- *True/false.* If A has a pivot in each column, then the rows of A are linearly independent.
- *True/false.* If u and v are vectors in $\mathbb{R}^5$ then $\{u, v, \sqrt{2}u - \pi v\}$ is linearly independent.
- If you have a set of linearly independent vectors, and their span is a line, how many vectors are in the set?