Parametric vector form and linear independence

Poll

Say you find the parametric vector form for a homogeneous system of linear equations, and you find that the set of solutions is the span of certain vectors. Then those vectors are...

1. always linearly independent
2. sometimes linearly independent
3. never linearly independent

Example. In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 = 8x_3 + 7x_4$
$x_2 = -4x_3 - 3x_4$
$x_3 = x_3$
$x_4 = x_4$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Parametric Vector Forms and Linear Independence

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In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors that appear are linearly independent (why?). This means that we can't write the solution with fewer than two vectors (why?). This also means that this way of writing the solution set is efficient: for each solution, there is only one choice of x_3 and x_4 that gives that solution.

Announcements Sep 16

- WeBWorK on Sections 2.4 and 2.5 due Thursday night

- First Midterm Friday 8 am - 8 pm on §1.1 - 2.5

- My office hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - ▶ Umar Fri 4:20-5:20
  - ▶ Seokbin Wed 10:30-11:30
  - ▶ Manuel Mon 5-6
  - ▶ Pu-ting Thu 2:30-3:30 (this week only)
  - ▶ Juntao Thu 3-4

Practice exam on Canvas.

- Review session Thu 3:30-5 with Umar

- No studio or office hours on Friday

- Tutoring: `http://tutoring.gatech.edu/tutoring`

- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`

- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`

- For general questions, post on Piazza

- Find a group to work with - let me know if you need help

- Counseling center: `https://counseling.gatech.edu`

Chapter 2

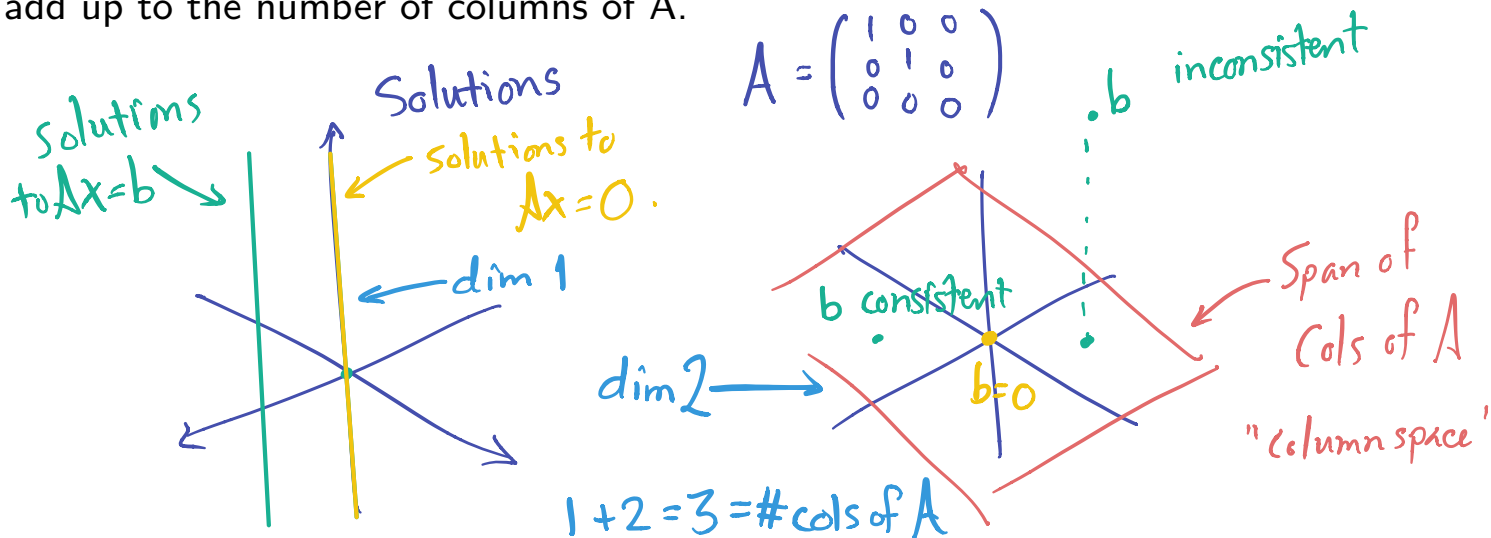## System of Linear Equations: Geometry

Where are we?

In Chapter 1 we learned to solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution. In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning. There are three main points:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Sec 2.3: $Ax = b$ is consistent $\Leftrightarrow$ b is in the span of the columns of A. ✓

Sec 2.4: The solutions to $Ax = b$ are parallel to the solutions to $Ax = 0$. ✓

Sec 2.9: The dim's of $\{b : Ax = b$ is consistent$\}$ and $\{$solutions to $Ax = b\}$ add up to the number of columns of A.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Solutions
to $Ax = b$

Solutions

Solutions to $Ax = 0$.

dim 1

dim 2 →

$1 + 2 = 3 = \#$ cols of A

$\cdot b$ inconsistent

b consistent

b=0

Span of Cols of A

"column space"

Section 2.6

Subspaces

Outline of Section 2.6

- Definition of subspace

- Examples and non-examples of subspaces

- Spoiler alert: Subspaces are the same as spans

- Spanning sets for subspaces

- Two important subspaces for a matrix: $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$

Subspaces

A subspace of $\mathbb{R}^n$ is a ⟨subset⟩ V of $\mathbb{R}^n$ with: *collection of pts in $\mathbb{R}^n$*

1. The zero vector is in V.

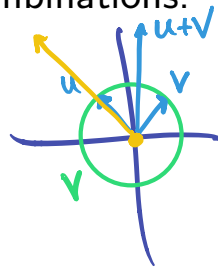2. If u and v are in V, then $u + v$ is also in V.

3. If u is in V and c is a scalar, then cu is in V.

The second and third properties are called "closure under addition" and "closure under scalar multiplication."

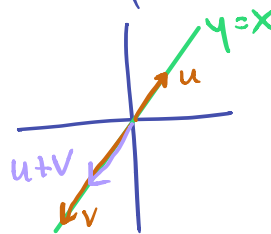Together, the second and third properties could together be rephrased as: closure under linear combinations.

Weird Example
$\{0\}$ is a subspace.

Example

1. ✗ $\Rightarrow$ *not a subsp.*
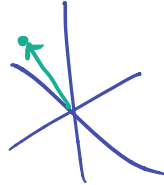2. ✗
3. ✗

Example

1. ✓
2. ✓
3. ✓

$\Rightarrow$ *subspace.*

Which are subspaces?

1. the unit circle in $\mathbb{R}^2$ — Last slide. No.

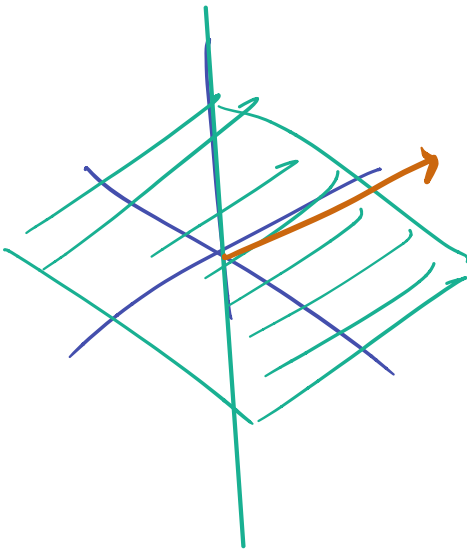2. the point $(1, 2, 3)$ in $\mathbb{R}^3$

   No. Fails 1, 2, 3

3. the xy-plane in $\mathbb{R}^3$

   Yes. ②$\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -10 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -9 \\ 0 \end{pmatrix}$ ③ $-7 \cdot \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -35 \\ -7 \\ 0 \end{pmatrix}$

4. the xy-plane together with the z-axis in $\mathbb{R}^3$

   No. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ — not in the set

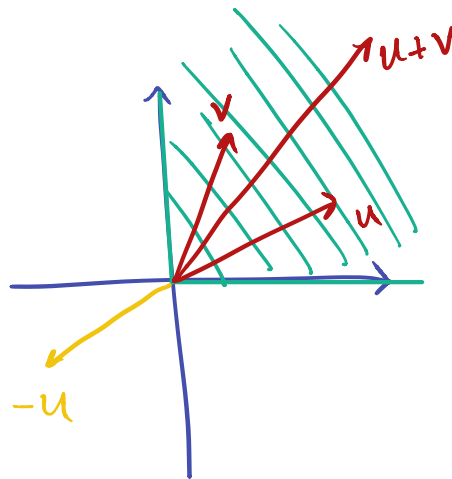   in xy-plane   in z-axis

   Fails 2.

   What about 1 & 3 ✓ ✓

Which are subspaces?

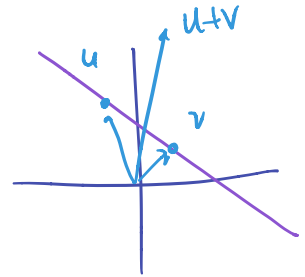Is the first quadrant of $\mathbb{R}^2$ a subspace?

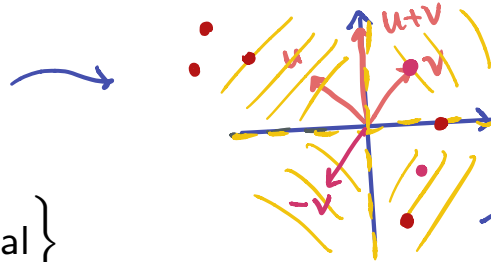1. yes
2. no

$u+v$

v

u

$-u$

1. ✓
2. ✓
3 ✗

No

Which are subspaces?

1. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \mid a + b = 0 \right\}$ yes — line through O.

2. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \mid a + b = 1 \right\}$ no — doesn't contain O (actually fails all 3)

3. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \mid ab \neq 0 \right\}$ No

everything
but axes.

Properties   1. ✗
of subsp   2. ✗
             3. ✗ mult. by
                    0.

4. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \mid a, b \text{ rational} \right\}$

No

Props
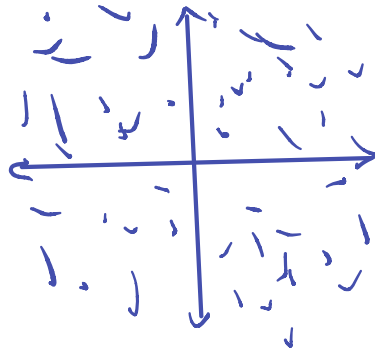1 ✓
2 ✓
3 ✗ $\sqrt{2} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}$

$\begin{pmatrix} \frac{51}{17} \\ \frac{3}{5} \end{pmatrix} + \begin{pmatrix} -\frac{117}{13} \\ \frac{5}{9} \end{pmatrix}$
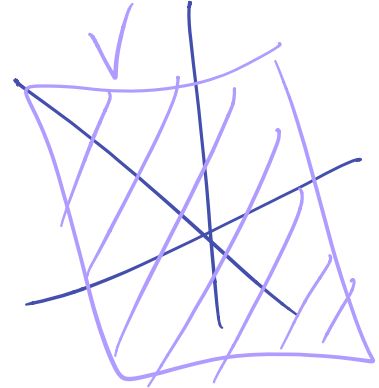
Spans and subspaces

Fact. Any $\mathrm{Span}\{v_1, \ldots, v_k\}$ is a subspace. ① $0 = 0 \cdot v_1 + \ldots + 0 \cdot v_k$

Why? ② $u = 5v_1 - v_2$
$\quad v = 2v_2 + v_3$ $\qquad u+v = 5v_1 + v_2 + v_3$ ③ Similar.

Fact. Every subspace V is a span.

Why? $\qquad V = $ span of all the vectors in V.

So now we know that three things are the same:

- subspaces
- spans
- planes through 0

So why bother with the word "subspace"? Sometimes easier to check a subset is a subspace than to check it is a span (see null spaces, eigenspaces). Also, it makes sense (and is often useful) to think of a subspace *without a particular spanning set in mind*.

Column Space and Null Space

$A = m \times n$ matrix.

$\mathrm{Col}(A) =$ column space of $A =$ span of the columns of A — row reduce.

$\mathrm{Nul}(A) =$ null space of $A =$ (set of solutions to $Ax = 0$)

Example. $A = \begin{pmatrix} \overset{x}{1} & \overset{y}{1} \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\mathrm{Col}(A) =$ line in $\mathbb{R}^3$

$\mathrm{Nul}(A) =$ line in $\mathbb{R}^2$

$1 + 1 = 2 = \#$ cols of A.

$\mathrm{Col}(A) =$ subspace of $\mathbb{R}^m$

$\mathrm{Nul}(A) =$ subspace of $\mathbb{R}^n$

We have already been interested in both. We have been computing null spaces all semester. Also, we have seen that $Ax = b$ is consistent exactly when b is in the span of the columns of A, or, b is in $\mathrm{Col}(A)$.

Spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

For $\mathrm{Nul}(A)$, row reduce $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$x = -y - z$
$y = y$
$z = z$

$$y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad ②$$

For $\mathrm{Col}(A)$, take cols of A (not reduced version!) corresponding to pivots

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad ①$$

$② + ① = 3 = \#$ cols.

Spanning sets for $\text{Nul}(A)$ and $\text{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a spanning set for $\text{Nul}(A)$
- the pivot columns of A form a spanning set for $\text{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

Notice that the columns of A form a (possibly larger) spanning set. We'll see later that the above recipe is the smallest spanning set.

Spanning sets

Find a spanning set for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

Null Space of

$$(2 \ 3 \ 1)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (0)$$

$$A = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 3/2 & 1/2 \end{pmatrix}$$

$$x = -3/2 y - 1/2 z$$

$$y = y$$

$$z = z$$

$$\left\{ \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Subspaces and Null spaces

Fact. Every subspace is a null space.

Why? Given a spanning set, you can reverse engineer the A...

It's actually a little tricky to do this. Given the spanning set, you make those vectors the rows of a matrix, then row reduce and find vector parametric form, and then make those vectors the rows of a new matrix. Why does this work? Try an example!

Example. Find a matrix A whose null space is the span of $(1, 1, 1)$ and $(1, 2, 3)$.

So now we know that four things are the same:

- subspaces
- spans
- planes through 0
- solutions to $Ax = 0$

So why learn about subspaces?

If subspaces are the same as spans, planes through the origin, and solutions to $Ax = 0$, why bother with this new vocabulary word?

The point is that we have been throwing around terms like "3-dimensional plane in $\mathbb{R}^4$" all semester, but we never said what "dimension" and "plane" are. Subspaces give the proper way to define a plane. Soon we will learn the meaning of a dimension of a subspace.

Section 2.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
    1. The zero vector is in V.
    2. If u and v are in V, then $u + v$ is also in V.
    3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.

- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form

- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)

- Four things are the same: subspaces, spans, planes through 0, null spaces

Typical exam questions

- Consider the set $\{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$. Is it a subspace? If not, which properties does it fail?

- Consider the x-axis in $\mathbb{R}^3$. Is it a subspace? If not, which properties does it fail?

- Consider the set $\{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\}$. Is it a subspace? If not, which properties does it fail?

- Find spanning sets for the column space and the null space of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- True/False: The set of solutions to a matrix equation is always a subspace.

- True/False: The zero vector is a subspace.