Announcements Sep 21

- WeBWorK on Section 2.6 due Thursday night
- Second Midterm Friday OCt 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- My office hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Regular studio on Friday
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Quiz Fri on 2.6

Chapter 2

System of Linear Equations: Geometry

Where are we?

In Chapter 1 we learned to solve any system of linear equations in any number of variables. The answer is row reduction, which gives an algebraic solution. In Chapter 2 we put some geometry behind the algebra. It is the geometry that gives us intuition and deeper meaning. There are three main points:

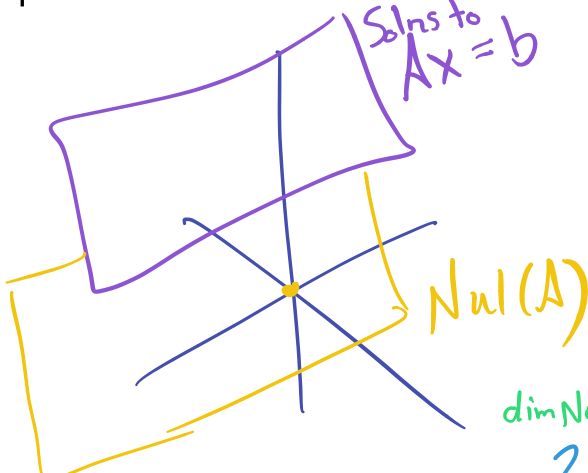
Sec 2.3: $Ax = b$ is consistent $\Leftrightarrow$ b is in the span of the columns of A. ✓

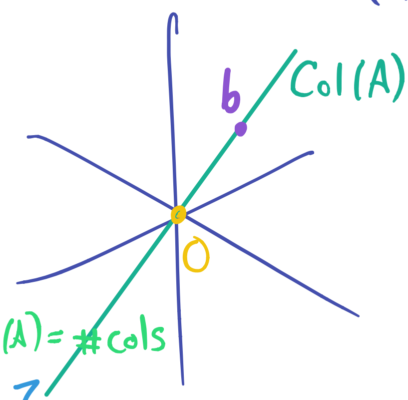
Sec 2.4: The solutions to $Ax = b$ are parallel to the solutions to $Ax = 0$. ✓

Sec 2.9: The dim's of $\{b : Ax = b \text{ is consistent}\}$ and $\{\text{solutions to } Ax = b\}$ add up to the number of columns of A.

$A = 3 \times 3$

$A = \begin{pmatrix} | & | & | \\ | & | & | \\ | & | & | \end{pmatrix}$





dim Nul(A) + dim Col(A) = # cols

$2 + 1 = 3$

Section 2.6

Subspaces

Section 2.6 Summary

A subset V of $\mathbb{R}^n$ is a subspace if: when we take any lin combo of vectors in V, we get another vector in V.

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u + v$ is also in V.
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.
- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- ~~Four~~ Five things are the same: subspaces, spans, planes through 0, null spaces column spaces.

$\mathrm{Nul}(A)$ = solns to $Ax = 0$

$\mathrm{Col}(A)$ = span of cols of A

Section 2.7

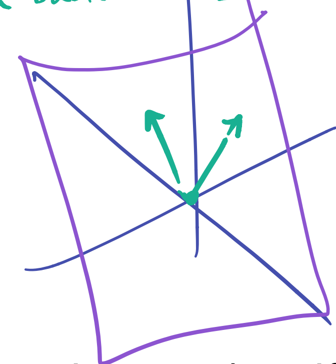
Bases

Bases

$V =$ subspace of $\mathbb{R}^n$

(For a point = origin, a basis has zero vectors)

For a 2D plane in $\mathbb{R}^3$, a basis has 2 vectors.



$k=2$
$n=3$

A basis for V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

1. $V = \text{Span}\{v_1, \ldots, v_k\}$
2. $v_1, \ldots, v_k$ are linearly independent

For a line in $\mathbb{R}^3$, basis has one vector. $k=1$, $n=3$

If $V = \mathbb{R}^n$, then $k=n$.

Equivalently, a basis is a *minimal spanning set*, that is, a spanning set where if you remove any one of the vectors you no longer have a spanning set.

$\dim(V) =$ dimension of $V = k =$ the number of vectors in the basis

(What is the problem with this definition of dimension?)

Need to know all bases have same # of vectors. But it's true.

Q. What is one basis for $\mathbb{R}^2$? $\mathbb{R}^n$? How many bases are there?

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ or $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}$

$\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$ "standard basis vectors"

$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

Basis example

Find a basis for the xy-plane in $\mathbb{R}^3$? Find all bases for the xy-plane in R^3. (Remember: a basis is a set of vectors in the subspace that span the subspace and are linearly independent.)

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

all of them: $\left\{ \begin{pmatrix} * \\ * \\ 0 \end{pmatrix}, \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \right\}$ not co-linear/ multiples.

Bases for $\mathbb{R}^n$

What are all bases for $\mathbb{R}^n$? So $V = \mathbb{R}^n$

Take a set of vectors $\{v_1, \dots, v_k\}$. Make them the columns of a matrix.

For the vectors to be linearly independent we need a pivot in every column. ✓

For the vectors to span $\mathbb{R}^n$ we need a pivot in every row. ✓

Conclusion: $k = n$ and the matrix has n pivots.

Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$?

No! Too many vectors.

Is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ a basis for $\mathbb{R}^3$? Rowreduce $\rightsquigarrow \begin{pmatrix} \square & 0 & 0 \\ 0 & \square & 0 \\ 0 & 0 & 0 \end{pmatrix}$

No! Only 2 pivots.

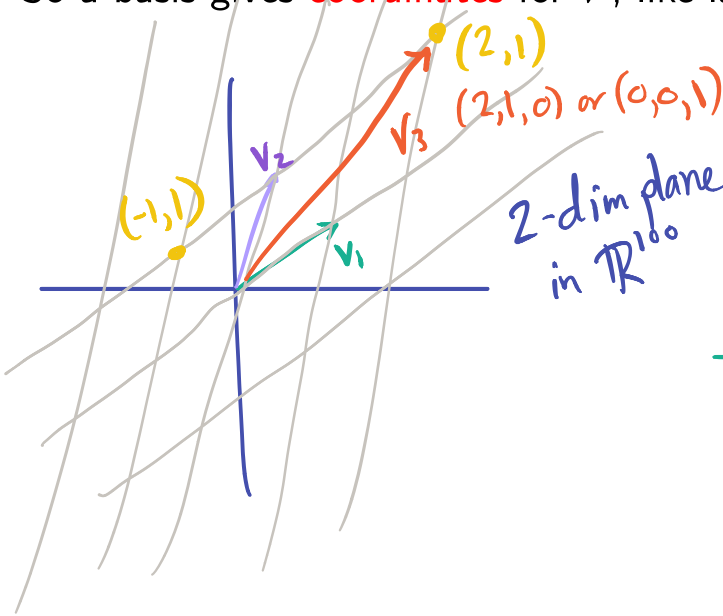
Who cares about bases

A basis $\{v_1, \ldots, v_k\}$ for a subspace V of $\mathbb{R}^n$ is useful because:

Every vector v in V can be written in exactly one way:

$$v = c_1 v_1 + \cdots + c_k v_k$$

So a basis gives coordinates for V, like latitude and longitude. See Section 2.8.



Too few vectors: not every pt has an address in terms of the basis.

Too many vectors: pts have ∞ many addresses.

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\mathrm{Nul}(A)$: $\rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{matrix} x = -y - z \\ y = y \\ z = z \end{matrix} \rightsquigarrow \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

plane in $\mathbb{R}^3$

$\mathrm{Col}(A)$: $\rightsquigarrow \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ So take first col of A $\rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

line in $\mathbb{R}^3$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{Nul}(A) \longrightarrow$ vect. param form, take those vectors.

$\mathrm{Col}(A) \longrightarrow \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a basis for $\mathrm{Nul}(A)$
- the pivot columns of A form a basis for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

column space

$\shortparallel$

What should you do if you are asked to find a basis for Span$\{v_1, \ldots, v_k\}$?

Make a matrix with cols $v_1, \ldots, v_k$.

do this

Bases for planes

Find a basis for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

· this is Nul(A)

where $A = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$.

See last slide: vect param form.

Basis theorem

Basis Theorem

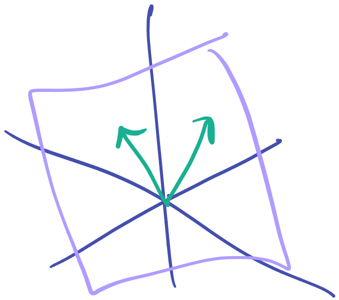
If V is a k-dimensional subspace of $\mathbb{R}^n$, then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V, linearly independent, k vectors

Think about $k=2$



We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

Section 2.7 Summary

- A basis for a subspace V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{Span}\{v_1, \ldots, v_k\}$
  2. $v_1, \ldots, v_k$ are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- Basis Theorem. Suppose V is a k-dimensional subspace of $\mathbb{R}^n$. Then
  - Any k linearly independent vectors in V form a basis for V.
  - Any k vectors in V that span V form a basis.

Typical exam questions

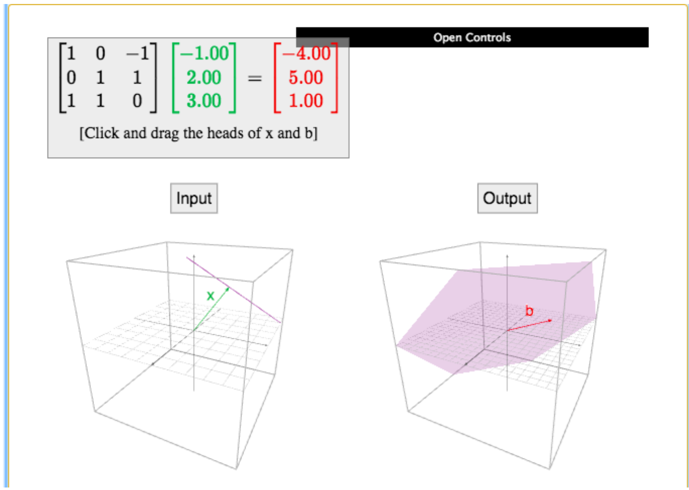
- Find a basis for the yz-plane in $\mathbb{R}^3$
- Find a basis for $\mathbb{R}^3$ where no vector has a zero
- How many vectors are there in a basis for a line in R^7?
- True/false: every basis for a plane in $\mathbb{R}^3$ has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in $\mathbb{R}^3$ and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of A is the number of pivots of A.
- True/false: If b lies in the column space of A, and the columns of A are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span R^3 must be linearly independent.

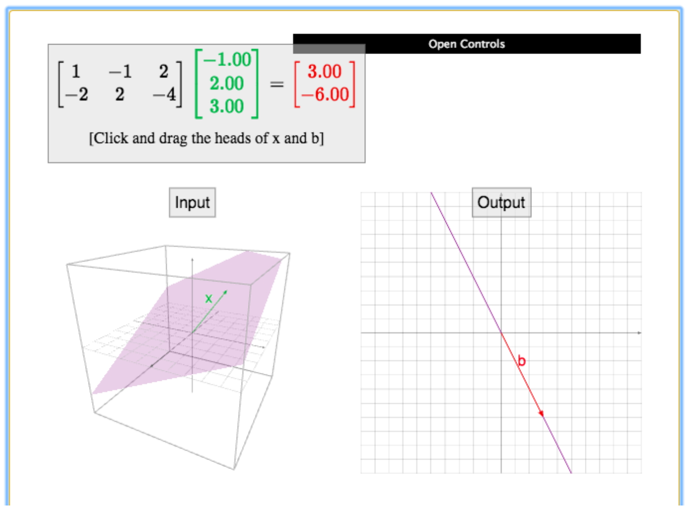
Section 2.9

The rank theorem

Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\mathrm{Col}(A)$:





Rank Theorem

$$\begin{aligned} \text{rank}(A) &= \dim \text{Col}(A) = \#\text{ pivot columns} \\ \text{nullity}(A) &= \dim \text{Nul}(A) = \#\text{ nonpivot columns} \end{aligned}$$

Rank Theorem. $\text{rank}(A) + \text{nullity}(A) = \#\text{cols}(A)$

This ties together everything in the whole chapter: rank A describes the b's so that $Ax = b$ is consistent and the nullity describes the solutions to $Ax = 0$. So more flexibility with b means less flexibility with x, and vice versa.

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Typical exam questions

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?

Section 2.9 Summary

- **Rank Theorem.** $\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = \#\mathrm{cols}(A)$