Announcements Sep 23

- WeBWorK on Section 2.6 due Thursday night
- Quiz on Section 2.6 Friday 8 am - 8 pm EDT
- My Office Hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Regular Studio on Friday
- Second Midterm Friday Oct 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Section 2.7

Bases

Bases

V = subspace of $\mathbb{R}^n$

A basis for V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

1. $V = \text{Span}\{v_1, \ldots, v_k\}$
2. $v_1, \ldots, v_k$ are linearly independent

Equivalently, a basis is a *minimal spanning set*, that is, a spanning set where if you remove any one of the vectors you no longer have a spanning set.

Q. What is one basis for $\mathbb{R}^2$? $\mathbb{R}^n$? How many bases are there?

Dimension

$V =$ subspace of $\mathbb{R}^n$

$\dim(V) =$ dimension of $V = k =$the number of vectors in the basis

(What is the problem with this definition of dimension?)

Basis theorem

Basis Theorem

If V is a k-dimensional subspace of $\mathbb{R}^n$, then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V, linearly independent, k vectors

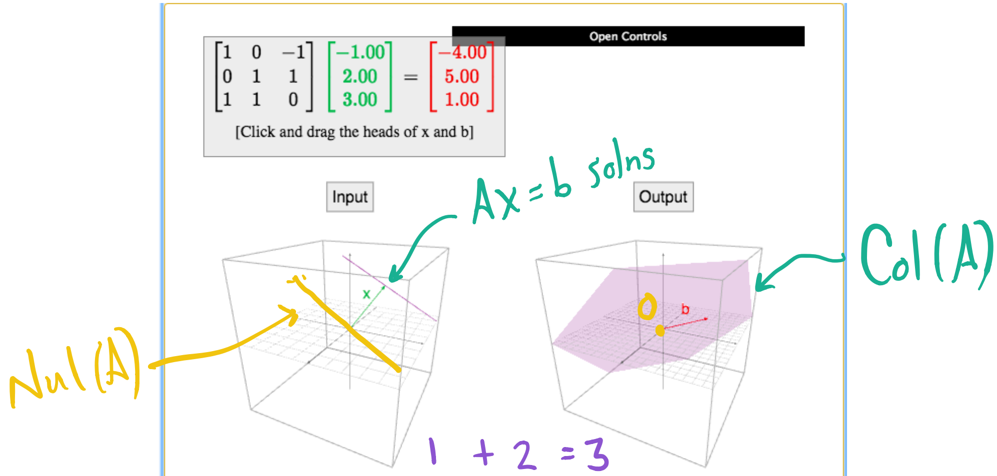
We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

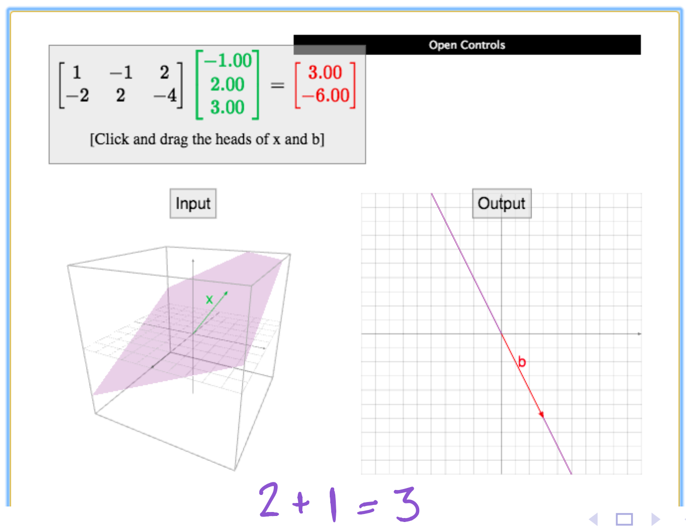
Section 2.9

The rank theorem

Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\mathrm{Col}(A)$:





Rank Theorem

$$\text{rank}(A) = \dim \text{Col}(A) = \#\ \text{pivot columns}$$
$$\text{nullity}(A) = \dim \text{Nul}(A) = \#\ \text{nonpivot columns}$$

Rank Theorem. $\text{rank}(A) + \text{nullity}(A) = \#\text{cols}(A)$

This ties together everything in the whole chapter: rank A describes the b's so that $Ax = b$ is consistent and the nullity describes the solutions to $Ax = 0$. So more flexibility with b means less flexibility with x, and vice versa.

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

About names

Again, why did we need all these vocabulary words? One answer is that the rank theorem would be harder to understand if it was:

The size of a minimal spanning set for the set of solutions to $Ax = 0$ plus the size of a minimal spanning set for the set of b so that $Ax = b$ has a solution is equal to the number of columns of A.

Compare to: $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$

"A common concept in history is that knowing the name of something or someone gives one power over that thing or person." –Loren Graham
`http://philoctetes.org/news/the_power_of_names_religion_mathematics`

Section 2.9 Summary

- **Rank Theorem.** $\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = \#\operatorname{cols}(A)$

Typical exam questions

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 5 dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?

Sections 3.1

Matrix Transformations

Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$

From matrices to functions

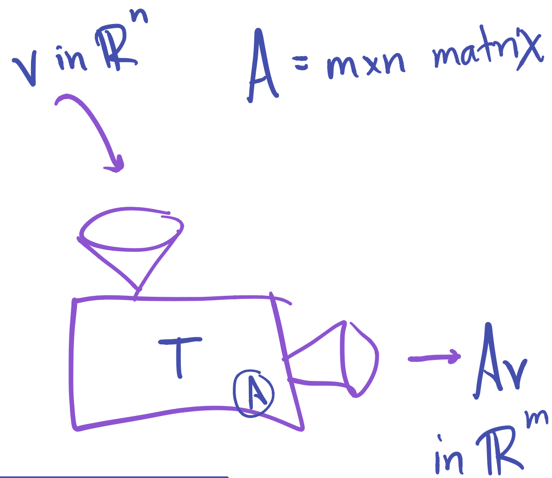
Let A be an $m \times n$ matrix.

A = m x n matrix

We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$T(v) = Av$$

This is called a matrix transformation.

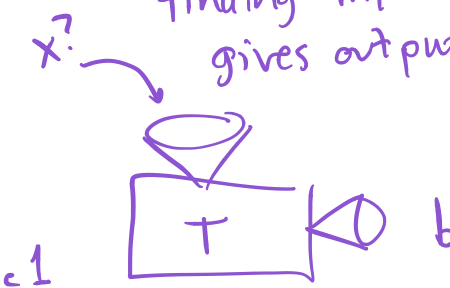


The domain of T is $\mathbb{R}^n$. all possible inputs

The co-domain of T is $\mathbb{R}^m$. all possible outputs

The range of T is the set of outputs: $\mathrm{Col}(A)$

Solving $Ax=b$ means finding input x that gives output b.



This gives us a*nother* point of view of $Ax = b$

example. range of $f(x)=x^2$ in Calc 1 is $[0,\infty)$. Codomain is $\mathbb{R}$

Demo

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$.

What is $T(u)$?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$$

Find v in $\mathbb{R}^2$ so that $T(v) = b$

Solving $Ax=b$!

$$v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Find (x,y) so

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$$

Reduce !

Find a vector in $\mathbb{R}^3$ that is not in the range of T.

any vector with different 1st & 3rd entries.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

since

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x+y \end{pmatrix}$$

Samesies

Square matrices

For a square matrix we can think of the associated matrix transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

as doing something to $\mathbb{R}^n$.

Example. The matrix transformation T for

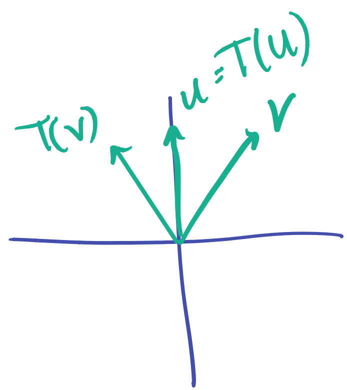
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Input : $\begin{pmatrix} x \\ y \end{pmatrix}$

Output : $\begin{pmatrix} -x \\ y \end{pmatrix}$

What does T do to $\mathbb{R}^2$?

Reflection about y-axis



Square matrices

What does [the matrix transf. corresp. to] each matrix do to $\mathbb{R}^2$?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Input: $\begin{pmatrix} x \\ y \end{pmatrix}$

Output: $\begin{pmatrix} y \\ x \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

Reflection about $y = x$.

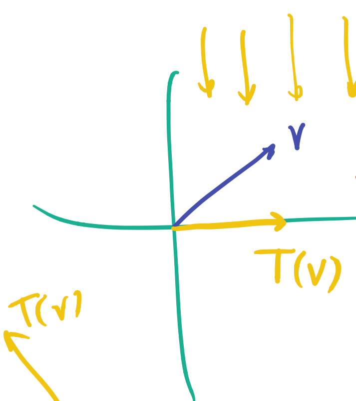
Range: $\mathbb{R}^2$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Input: $\begin{pmatrix} x \\ y \end{pmatrix}$

Output: $\begin{pmatrix} x \\ 0 \end{pmatrix}$

(Orthogonal) Projection to x-axis



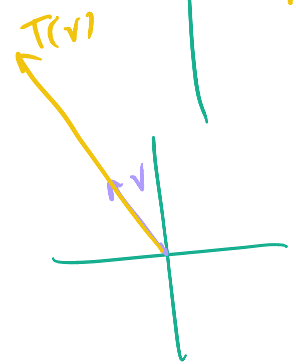
Range: x-axis

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Input: $\begin{pmatrix} x \\ y \end{pmatrix}$

Output: $\begin{pmatrix} 3x \\ 3y \end{pmatrix}$

Dilation by 3



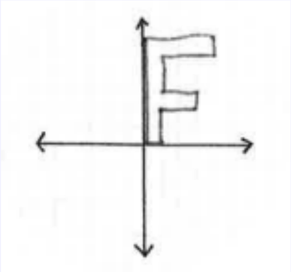
Range: $\mathbb{R}^2$

What is the range in each case?

Poll

Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?



Square matrices

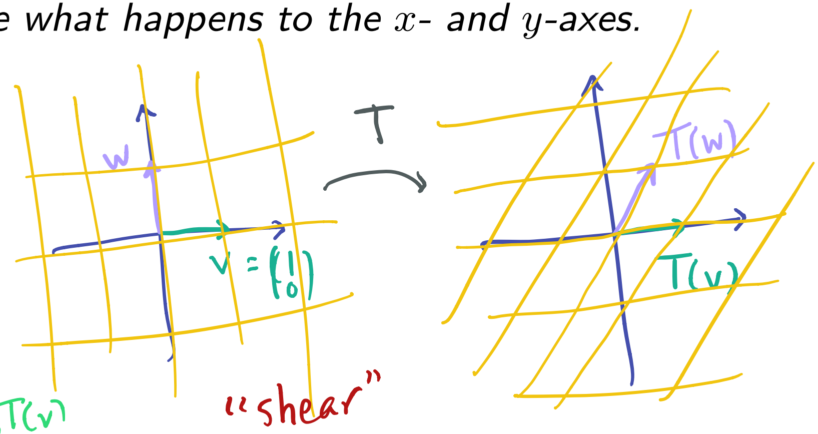
What does each matrix do to $\mathbb{R}^2$?

Hint: if you can't see it all at once, see what happens to the x- and y-axes.

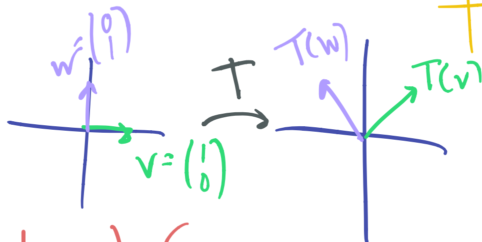
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Input : $\begin{pmatrix} x \\ y \end{pmatrix}$

Output : $\begin{pmatrix} x+y \\ y \end{pmatrix}$



$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$



"rotation by $\pi/4$ + dilation by $\sqrt{2}$"

θ = some fixed number

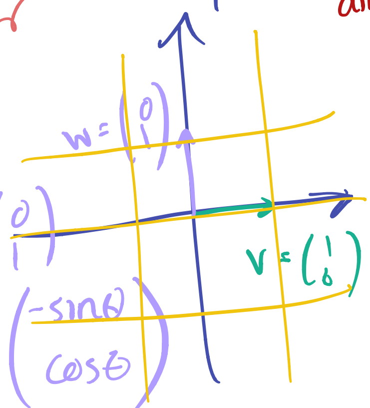
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

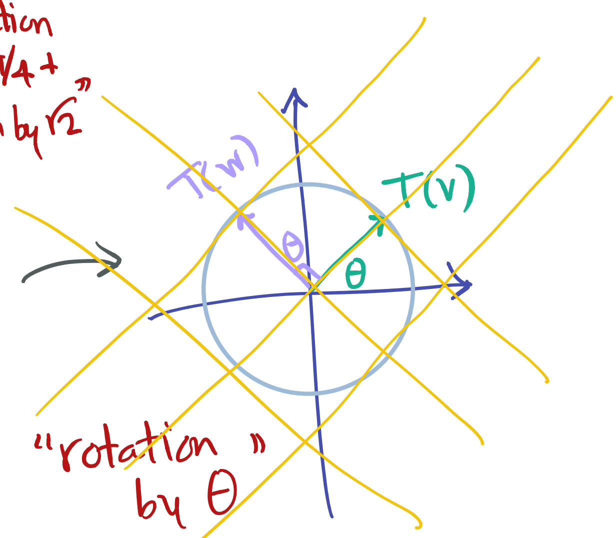
Input : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Output : $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

Input : $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Output : $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$





"rotation by θ"

Examples in $\mathbb{R}^3$

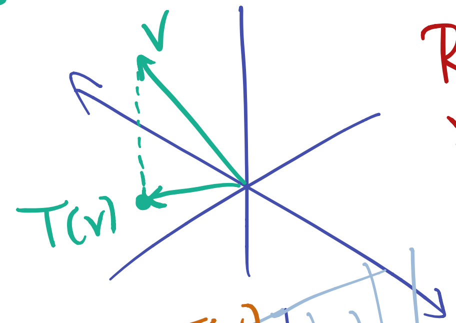
What does each matrix do to $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Input: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Output: $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

Orthogonal Projection to xy plane



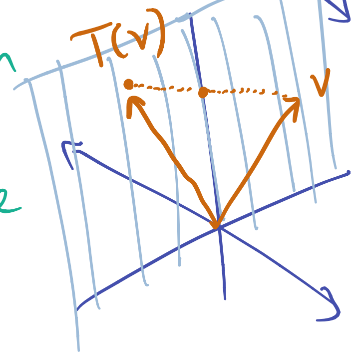
Range: xy-plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Input: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

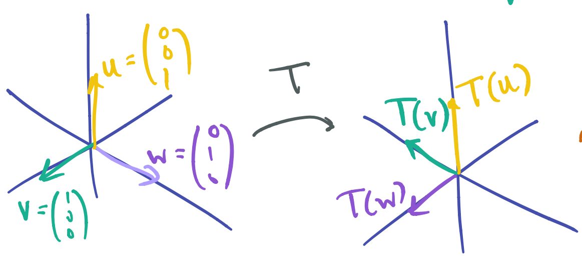
Output: $\begin{pmatrix} x \\ -y \\ z \end{pmatrix}$

Reflection about xz-plane



Range: $\mathbb{R}^3$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Rotation by 90° about z-axis

Range: $\mathbb{R}^3$

Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

First day slides

Key example: Rabbits

1/2 first years survive

1/2 second yrs survive

2nd have 6 babies

3rd yrs have 8 babies

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

Input: Population in a given year (#1st years, #2nd, #3rd)

Output: Population for following year

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} F \\ S \\ T \end{pmatrix} = \begin{pmatrix} 6S+8T \\ 1/2 F \\ 1/2 S \end{pmatrix}$$

~~$$\begin{pmatrix} 0 & 6 & 8 \\ 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}\begin{pmatrix} F \\ S \\ T \end{pmatrix} = \begin{pmatrix} 6S+8T \\ 1/2 S \\ 1/2 F \end{pmatrix}$$~~

No!

Typical exam questions

- What does the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ do to $\mathbb{R}^3$?
- What does the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- True/false. If A is a matrix and T is the associated matrix transformation, then the statement $Ax = b$ is consistent is equivalent to the statement that b is in the range of T.
- True/false. There is a matrix A so that the domain of the associated matrix transformation is a line in $\mathbb{R}^3$.

Sections 3.2

One-to-one and onto transformations

Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution
- A has a pivot in each column
- the range of T has dimension n

What can we say about the relative sizes of m and n if T is one-to-one?

Draw a picture of the range of a one-to-one matrix transformation $\mathbb{R} \to \mathbb{R}^3$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is onto
- the columns of A span $\mathbb{R}^m$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

What can we say about the relative sizes of m and n if T is onto?

Give an example of an onto matrix transformation $\mathbb{R}^3 \to \mathbb{R}$.

One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

$$\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ reflection

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$ projection

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$ scaling

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ shear

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ rotation