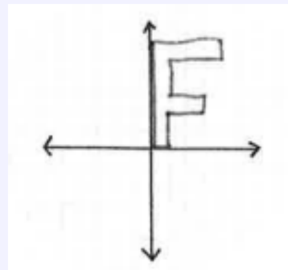
Poll

Polls channel in Teams

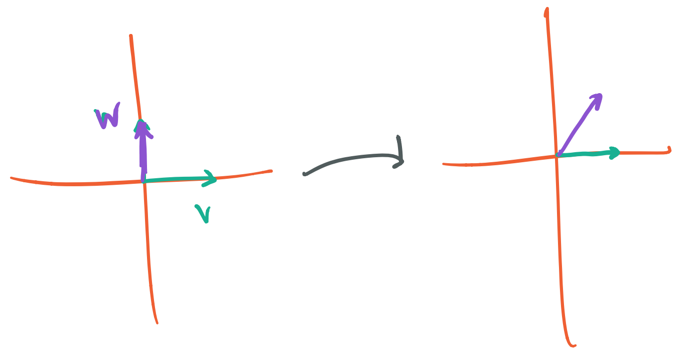
Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$$

"shear"



Announcements Sep 28

- WeBWorK on Sections 2.7, 2.9, 3.1 due Thursday night
- Quiz on Section 2.7, 2.9, 3.1 Friday 8 am - 8 pm EDT
- My Office Hours ~~Tue 11-12~~, Thu 1-2, and by appointment
- TA Office Hours need to change – check email.
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Second Midterm Friday Oct 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: `https://counseling.gatech.edu`

Sections 3.1

Matrix Transformations

Chapter 3: reframing Chaps 1 & 2 in terms of algebra

From matrices to functions

Let A be an $m \times n$ matrix.

We define a function ← rule with inputs & outputs. each input has only one output

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(v) = Av$$

This is called a matrix transformation.

" function

The domain of T is $\mathbb{R}^n$. inputs

The co-domain of T is $\mathbb{R}^m$. possible outputs

The range of T is the set of outputs: $\mathrm{Col}(A)$

This gives us a*nother* point of view of $Ax = b$

Solving Ax=b is finding input x with output b for T

Demo

Why are we learning about matrix transformations?

Sample applications:

- Cryptography (Hill cypher)
- Computer graphics (Perspective projection is a linear map!)
- Aerospace (Control systems - input/output)
- Biology
- Many more!

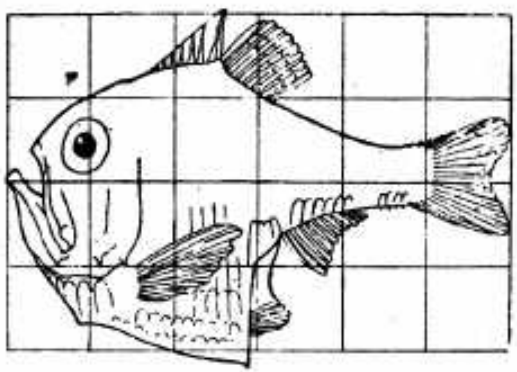


Fig. 517. *Argyropelecus Olfersi.*

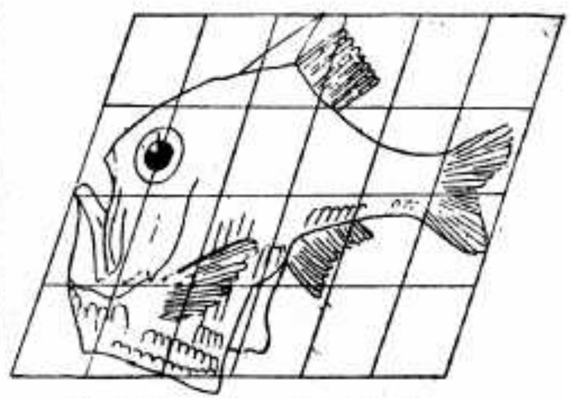


Fig. 518. *Sternoptyx diaphana.*

Applications of Linear Algebra

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

1st, 2nd, 3rd yr rabbits in one year

These relations can be represented using a matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 92 \\ 2 \\ 3 \end{pmatrix}$$

input — output — next year.

How does this relate to matrix transformations?

Demo

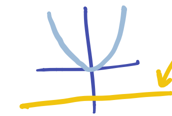
Section 3.2

One-to-one and onto transformations

Section 3.2 Outline

- Learn the definitions of one-to-one and onto functions
- Determine if a given matrix transformation is one-to-one and/or onto

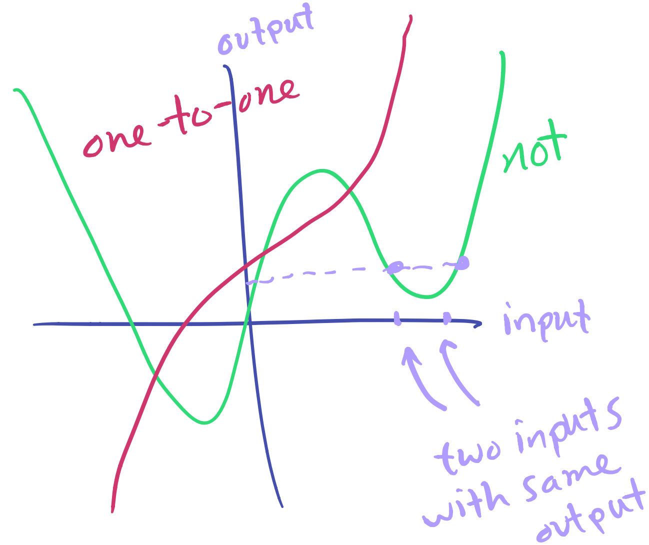
example $f(x) = x^2$ not onto.
-1 is not in the range

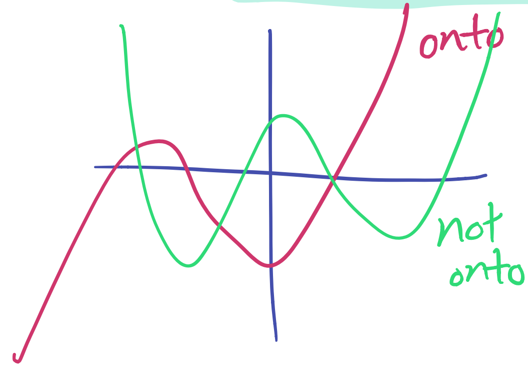


In Calculus:

One-to-one
$\longleftrightarrow$ horizontal line test (each horizontal line cross graph ≤ 1 pt.)
$\longleftrightarrow$ each input has one output

Onto
$\longleftrightarrow$ each horizontal line crosses graph ≥ 1 pt
$\longleftrightarrow$ all possible outputs are actual outputs
codomain = range





One-to-one

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs.

Do not confuse this with the definition of a function, which says that for each input x in $\mathbb{R}^n$ there is at most one output b in $\mathbb{R}^m$.

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

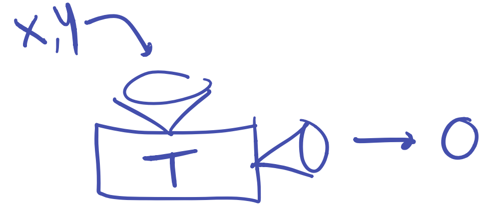
Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution
- A has a pivot in each column
- the range of T has dimension n

Col(A)

already said these are same

Why is one-to-one same as $Ax=0$ has only 0 soln?

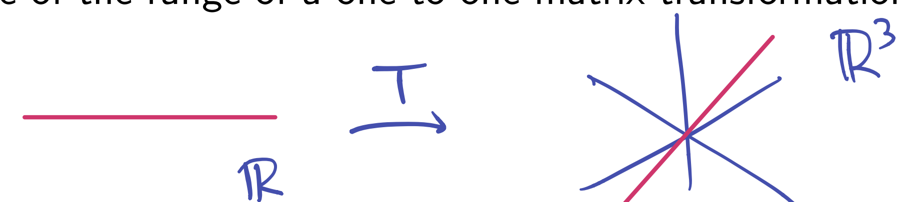


What can we say about the relative sizes of m and n if T is one-to-one?

$m \geq n$ tall or square, not wide

Draw a picture of the range of a one-to-one matrix transformation $\mathbb{R} \to \mathbb{R}^3$.

3×1 matrix



If T was not one-to-one, range would be a pt.

Onto

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

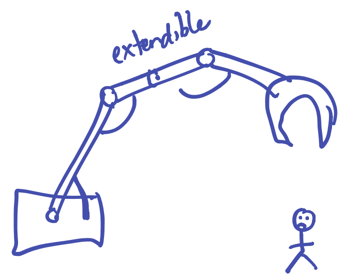
Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is onto
- the columns of A span $\mathbb{R}^m$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

direct rephrasing

already said these are same.

Read about robot arms in ILA



$\text{Col}(A) = \mathbb{R}^m$

• Rows lin ind.

What can we say about the relative sizes of m and n if T is onto?

$m \le n$ wide or square, not tall

Give an example of an onto matrix transformation $\mathbb{R}^3 \to \mathbb{R}^2$.

2×3 projection

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

$$\begin{pmatrix} \boxed{1} & 0 & 7 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{9} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

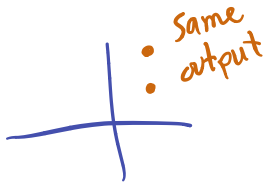
one-to-one	✓	✓ cols lin ind	✗	✓ cols lin ind	✗
onto	✓	✗ tall	✓	✓	✗

By the way: onto ⟷ rows lin ind.

One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

Matrix	Transformation	One-to-one	Onto
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	reflection about y=x	✓	✓
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	projection onto x-axis	✗	✗
$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	scaling	✓	✓
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	shear	✓	✓
$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	rotation	✓	✓



Which are one to one / onto?

$f(x) = x^3$ is one-to-one

Poll

Which give one to one-to-one / onto matrix transformations?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

not one-to-one

Like $f(x)=x^2$

Inputs: 2, -2

Same output: 4

not one-to-one

Demo

Demo

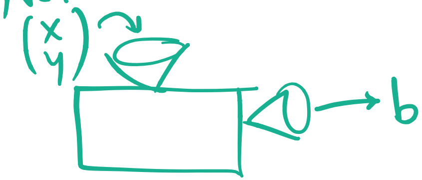
Demo

Find 2 inputs with same output for

Inputs: $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

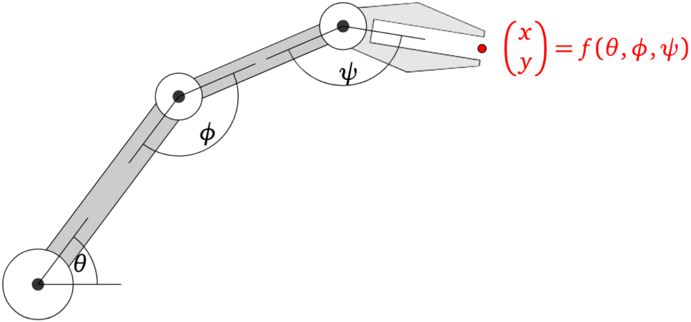
Same Output $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Not one-to-one



Robot arm

Consider the robot arm example from the book.



There is a natural function f here (not a matrix transformation). The input is a set of three angles and the co-domain is $\mathbb{R}^2$. Is this function one-to-one? Onto?

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are linearly independent
  - $Ax = 0$ has only the trivial solution
  - A has a pivot in each column
  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in each row
  - $Ax = b$ is consistent for all b in $\mathbb{R}^m$.
  - the range of T has dimension m

Typical exam questions

- True/False. It is possible for the matrix transformation for a 5×6 matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by projection to the yz-plane is onto.
- True/False. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by π is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.

Section 3.3

Linear Transformations

Section 3.3 Outline

- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Find the matrix for a linear transformation

Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

Like: lin. combos & subspaces

First examples: matrix transformations.

$$A(u+v) = Au + Av$$

$$A(c \cdot u) = cAu$$

Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

Notice that $T(0) = 0$. *Why?*

We have the standard basis vectors for $\mathbb{R}^n$:

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \end{aligned}$$

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. *Why?*

In engineering, this is called the principle of superposition.

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix A so that

$$T(v) = Av$$

for all v in $\mathbb{R}^n$.

The matrix for a linear transformation is called the standard matrix.

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

Why? Notice that $Ae_i = T(e_i)$ for all i. Then it follows from linearity that $T(v) = Av$ for all v.

The identity

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$T(v) = v$$

What is the standard matrix?

This standard matrix is called I_n or I.

Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

What is the standard matrix for T?

In fact, a function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables, no constant terms).

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the x-direction and 3 in the y-direction, and then reflects over the line $y = x$.

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.

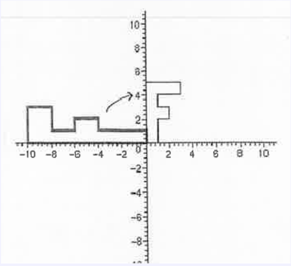
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion

Discussion Question

Find a matrix that does this.



Summary of 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.