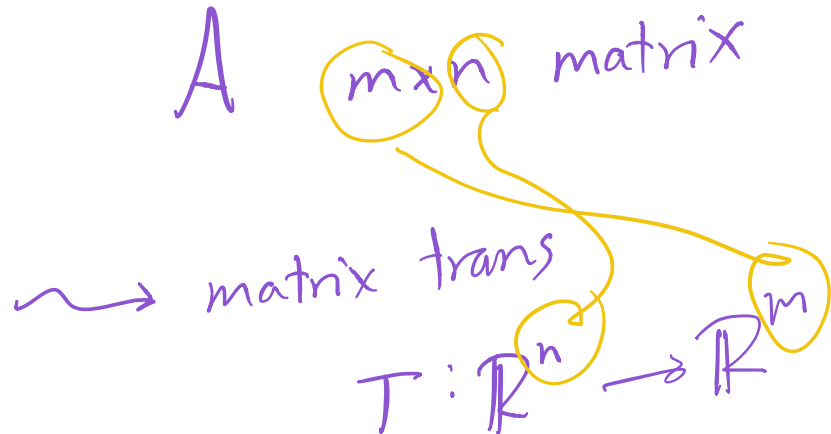# Announcements Sep 30

- WeBWorK on Sections 2.7, 2.9, 3.1 due Thursday night
- Quiz on Section 2.7, 2.9, 3.1 Friday 8 am - 8 pm EDT
- My Office Hours Tue 11-12, Thu 1-2, and by appointment
- TA Office Hours
  - Umar Fri 4:20-5:20
  - Seokbin Wed 10:30-11:30
  - Manuel Mon 5-6
  - Pu-ting Thu 3-4
  - Juntao Thu 3-4
- Studio on Friday
- Second Midterm Friday Oct 16 8 am - 8 pm on §2.6-3.6 (not §2.8)
- Tutoring: http://tutoring.gatech.edu/tutoring
- PLUS sessions: http://tutoring.gatech.edu/plus-sessions
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- For general questions, post on Piazza
- Find a group to work with - let me know if you need help
- Counseling center: https://counseling.gatech.edu

Chapter 3

Linear Transformations and Matrix Algebra

A $m \times n$ matrix

$\rightsquigarrow$ matrix trans

$T : \mathbb{R}^n \to \mathbb{R}^m$

Where are we?

In Chapter 1 we learned to solve all linear systems algebraically.

In Chapter 2 we learned to think about the solutions geometrically.

In Chapter 3 we continue with the algebraic abstraction. We learn to think about solving linear systems in terms of inputs and outputs. This is similar to control systems in AE, objects in computer programming, or hot pockets in a microwave.

More specifically, we think of a matrix as giving rise to a function with inputs and outputs. Solving a linear system means finding an input that produces a desired output. We will see that sometimes these functions are invertible, which means that you can reverse the function, inputting the outputs and outputting the inputs.

The invertible matrix theorem is the highlight of the chapter; it tells us when we can reverse the function. As we will see, it ties together everything in the course.

Section 3.2

One-to-one and onto transformations

One-to-one

function

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs.

Do not confuse this with the definition of a function, which says that for each input x in $\mathbb{R}^n$ there is ~~at most~~ one output b in $\mathbb{R}^m$.

exactly

$$\sqrt{\ } : [0, \infty) \longrightarrow \mathbb{R}$$

$$\text{or } \sqrt{\ } : [0, \infty) \longrightarrow [0, \infty)$$

$$\text{not } \sqrt{\ } : \mathbb{R} \longrightarrow \mathbb{R}$$

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution
- A has a pivot in each column
- the range of T has dimension n

What can we say about the relative sizes of m and n if T is one-to-one?

Draw a picture of the range of a one-to-one matrix transformation $\mathbb{R} \to \mathbb{R}^3$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is onto
- the columns of A span $\mathbb{R}^m$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

$$A \begin{pmatrix} \\ \end{pmatrix} = \begin{pmatrix} \\ \end{pmatrix}$$

$$\underset{7 \times 5}{m \times n} \quad \underset{5 \times 1}{n \times 1} \qquad \underset{7 \times 1}{m \times 1}$$

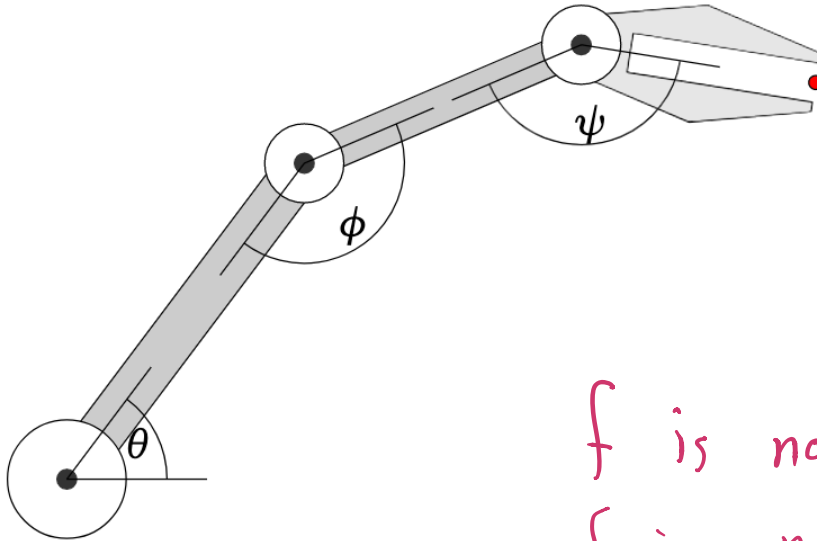What can we say about the relative sizes of m and n if T is onto?

$$n \geqslant m \qquad T : \mathbb{R}^5 \to \mathbb{R}^7$$
$$\text{never onto}$$

Give an example of an onto matrix transformation $\mathbb{R}^3 \to \mathbb{R}$.

$$\begin{pmatrix} \ddots \\ \end{pmatrix} \quad \mathbb{R}^7 \to \mathbb{R}^5$$
$$\text{sometimes onto}$$

Robot arm

Consider the robot arm example from the book.

input: triple of angles

$$\downarrow\downarrow\downarrow$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = f(\theta, \phi, \psi)$$

output: point in $\mathbb{R}^2$

ψ

ϕ

θ

f is not one-to-one

f is not onto.

There is a natural function f here (not a matrix transformation). The input is a set of three angles and the co-domain is $\mathbb{R}^2$. Is this function one-to-one? Onto?

Section 3.3

Linear Transformations

Section 3.3 Outline

- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Find the matrix for a linear transformation

Linear transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

First examples: matrix transformations. **Why?**

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$

5

$5 \cdot f(1) \neq f(5 \cdot 1) = 125$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boxed{xy} + y \\ x - y \end{pmatrix}$ not linear

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix}$ linear

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + 1 \\ x - y \end{pmatrix}$

not linear

$T(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0.$

$u + v$ u v

T
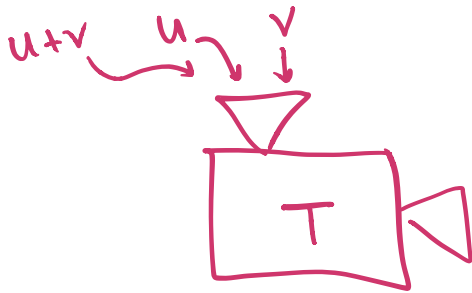
$T(u) = Au$
$T(v) = Av$
$T(u+v) = A(u+v)$

But $A(u+v) = Au + Av$

$\begin{pmatrix} & A & \\ & & \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ $\begin{pmatrix} A \end{pmatrix} \begin{pmatrix} u + v \end{pmatrix}$

Which are linear transformations?
And why?

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \\ x - y \end{pmatrix}$$

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + 1 \\ y \\ x - y \end{pmatrix}$$

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ y \\ x - y \end{pmatrix}$$

A function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables, no constant terms).

Linear transformations

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \mathbb{R}^5$

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if

- $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and c in $\mathbb{R}$.

Notice that $T(0) = 0$. *Why?* $\quad T(0) = T(0 \cdot v) = 0 \cdot T(v) = 0$

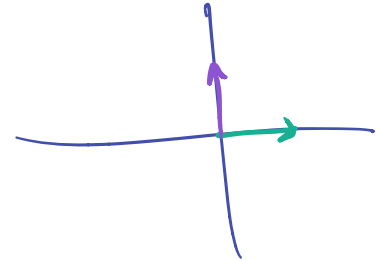We have the standard basis vectors for $\mathbb{R}^n$:

$\mathbb{R}^m$

$\mathbb{R}^n$

$$e_1 = (1, 0, 0, \ldots, 0)$$
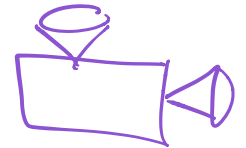$$e_2 = (0, 1, 0, \ldots, 0)$$
$$\vdots$$
$$e_n = (0, \ldots, 0, 1)$$

If we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$. *Why?*

$$T(5e_1 + 7e_2) = 5T(e_1) + 7T(e_2)$$

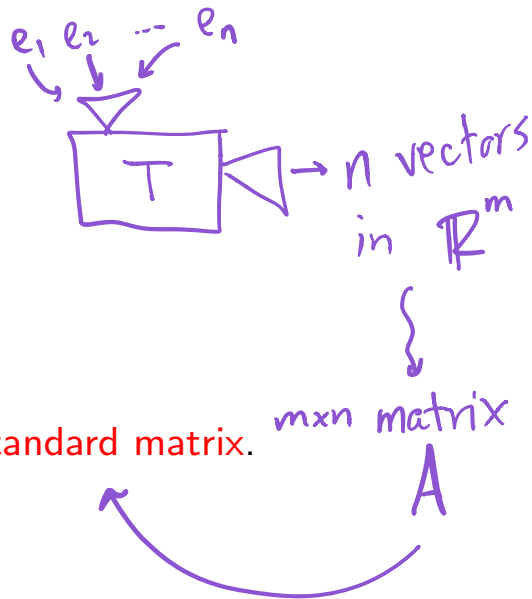In engineering, this is called the principle of superposition.

Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

This means that for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ there is an $m \times n$ matrix A so that

$$T(v) = Av$$

for all v in $\mathbb{R}^n$.

$e_1 \; e_2 \; \cdots \; e_n$

T

$\to n$ vectors in $\mathbb{R}^m$

$m \times n$ matrix A

The matrix for a linear transformation is called the standard matrix.
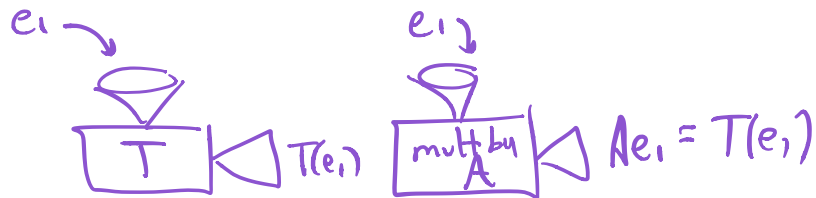
Linear transformations are matrix transformations

Theorem. Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$

1st col of A is $T(e_1)$ etc.

Why? Notice that $Ae_i = T(e_i)$ for all i. Then it follows from linearity that $T(v) = Av$ for all v.

e_1

e_1

T $T(e_1)$ mult by A $Ae_1 = T(e_1)$

The identity

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is

$$T(v) = v$$
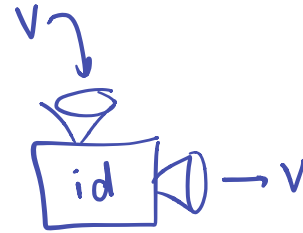
What is the standard matrix?

This standard matrix is called I_n or I.

$$n = 4$$

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{identity} \\ \text{matrix.}$$

$$I v = v \qquad \text{any } V.$$

Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ y \\ x - y \end{pmatrix}$$

or, take coefficients.

What is the standard matrix for T?

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \rightsquigarrow \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$
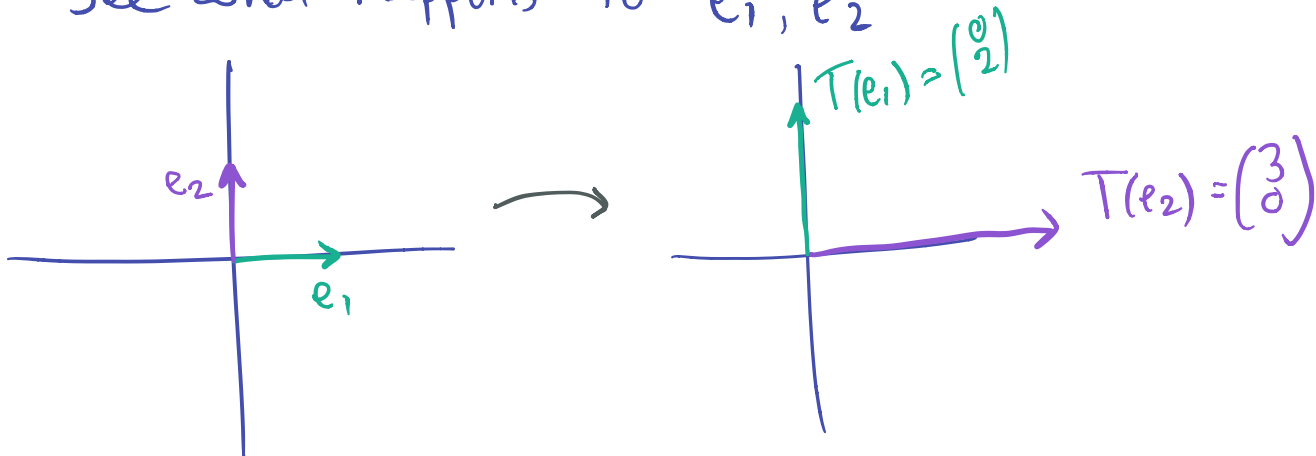
Check $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$

In fact, a function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables, no constant terms).

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the x-direction and 3 in the y-direction, and then reflects over the line $y = x$.

See what happens to e_1, e_2

$T(e_1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

e_2

e_1

$T(e_2) = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$

$T\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ 2x \end{pmatrix}$

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.
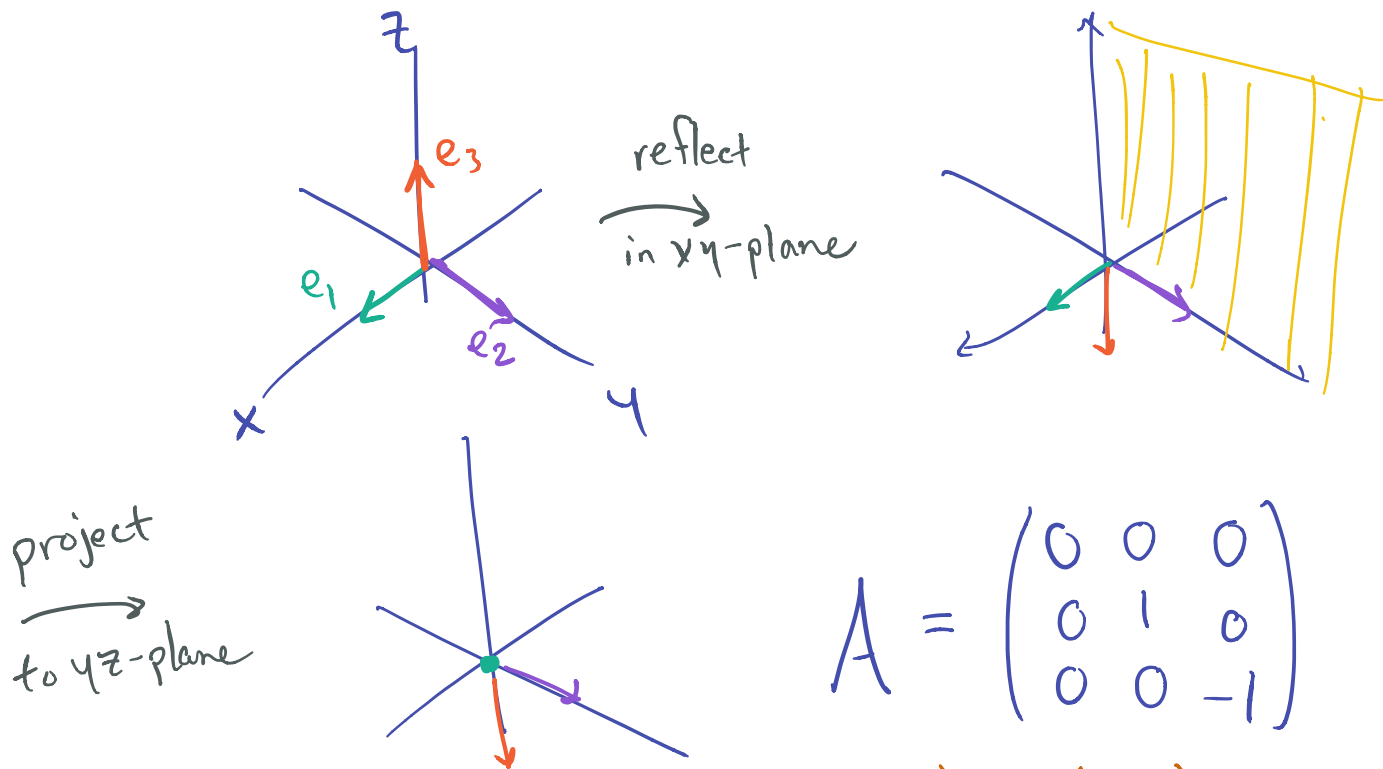
proj

to y-axis
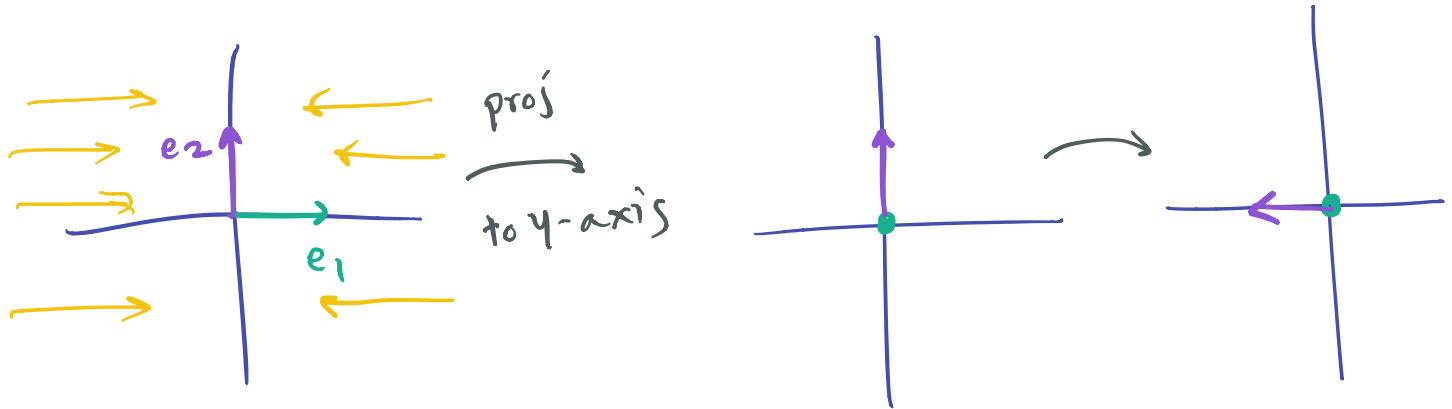
$$A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ 0 \end{pmatrix}$$

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.
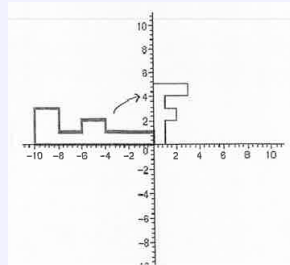

reflect
in xy-plane

project
to yz-plane

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ -z \end{pmatrix}$$

Discussion

Discussion Question

Find a matrix that does this.

Summary of Section 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - ▶ $T(u + v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
  - ▶ $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.

- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).

- The standard matrix for a linear transformation has its ith column equal to $T(e_i)$.

$$A \xrightarrow[\text{reduce}]{\text{row}} B$$

pivot cols of B

basis for $Col(A)$

Typical Exam Questions Section 3.3

- Is the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?
- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and that

$$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

What is

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix}?$$

- Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates about the z-axis by π and then scales by 2.
- Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by:

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ x \end{pmatrix}$$

Is this a linear transformation? If so, what is the standard matrix for T?

- Is the identity transformation one-to-one?