

**Supplemental problems: §1.2, §1.3**

1. Is the matrix below in reduced row echelon form?

$$\left( \begin{array}{cccc|c} 1 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

**Solution.**

Yes.

2. Put an augmented matrix into reduced row echelon form to solve the system

$$x_1 - 2x_2 - 9x_3 + x_4 = 3$$

$$4x_2 + 8x_3 - 24x_4 = 4.$$

**Solution.**

$$\left( \begin{array}{cccc|c} 1 & -2 & -9 & 1 & 3 \\ 0 & 4 & 8 & -24 & 4 \end{array} \right) \xrightarrow{R_2 = \frac{R_2}{4}} \left( \begin{array}{cccc|c} 1 & -2 & -9 & 1 & 3 \\ 0 & 1 & 2 & -6 & 1 \end{array} \right) \xrightarrow{R_1 = R_1 + 2R_2} \left( \begin{array}{cccc|c} \boxed{1} & 0 & -5 & -11 & 5 \\ 0 & \boxed{1} & 2 & -6 & 1 \end{array} \right)$$

The third and fourth columns are not pivot columns, so  $x_3$  and  $x_4$  are free variables.

Our equations are

$$x_1 - 5x_3 - 11x_4 = 5$$

$$x_2 + 2x_3 - 6x_4 = 1.$$

Therefore,

$$x_1 = 5 + 5x_3 + 11x_4$$

$$x_2 = 1 - 2x_3 + 6x_4$$

$$x_3 = x_3 \quad (\text{any real number})$$

$$x_4 = x_4 \quad (\text{any real number})$$

3. a) Row reduce the following matrices to reduced row echelon form.  
 b) If these are augmented matrices for a linear system (with the last column being after the = sign), then which are inconsistent? Which have a *unique* solution?

**Solution.**

$$\begin{array}{l} \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{array} \right) \\ \begin{array}{l} \xrightarrow{R_2 = R_2 - 4R_1} \\ \xrightarrow{R_3 = R_3 - 6R_1} \end{array} \end{array} \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{array} \right)$$

$$\left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{array} \right)$$

$$\begin{array}{l} R_2 = R_2 \div -3 \\ \hline \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 + 5R_2 \\ \hline \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} R_1 = R_1 - 2R_2 \\ \hline \end{array} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$\begin{array}{rcl} x & - & z = -2 \\ & y + 2z & = 3 \\ & 0 & = 0. \end{array}$$

This system is consistent, but since  $z$  is a free variable, it does not have a *unique* solution.

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 = R_2 - 3R_1 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 - 5R_1 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\begin{array}{l} R_2 = R_2 \div -4 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 + 8R_2 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 \div -10 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_1 = R_1 - 7R_3 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_2 = R_2 - 3R_3 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_1 = R_1 - 3R_2 \\ \hline \end{array} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it

corresponds to the system of linear equations

$$\begin{aligned}x - z &= 0 \\y + 2z &= 0 \\0 &= 1,\end{aligned}$$

which is inconsistent.

$$\begin{aligned}\begin{pmatrix} 3 & -4 & 2 & 0 \\ -8 & 12 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} & \xrightarrow{R_2 = R_2 + 3R_1} \begin{pmatrix} 3 & -4 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \\ & \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 3 & -4 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \\ & \xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \\ & \xrightarrow{R_3 = R_3 + 6R_1} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix} \\ & \xrightarrow{R_2 = R_2 \div -4} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix} \\ & \xrightarrow{R_3 = R_3 - 8R_2} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \\ & \xrightarrow{R_3 = R_3 \div 3} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ & \xrightarrow{R_1 = R_1 - 2R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ & \xrightarrow{R_2 = R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\end{aligned}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$x = 0 \quad y = 0 \quad z = 0,$$

which has a unique solution.

4. We can use linear algebra to find a polynomial that fits given data, in the same way that we found a circle through three specified points in the §1.1 Webwork.

Is there a degree-three polynomial  $P(x)$  whose graph passes through the points  $(-2, 6)$ ,  $(-1, 4)$ ,  $(1, 6)$ , and  $(2, 22)$ ? If so, how many degree-three polynomials have a graph through those four points? We answer this question in steps below.

- a) If  $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$  is a degree-three polynomial passing through the four points listed above, then  $P(-2) = 6$ ,  $P(-1) = 4$ ,  $P(1) = 6$ , and  $P(2) = 22$ . Write a system of four equations which we would solve to find  $a_0$ ,  $a_1$ ,  $a_2$ , and  $a_3$ .
- b) Write the augmented matrix to represent this system and put it into reduced row-echelon form. Is the system consistent? How many solutions does it have?

### Solution.

- a) We compute

$$\begin{aligned} P(-2) = 6 &\implies a_0 + a_1 \cdot (-2) + a_2 \cdot (-2)^2 + a_3 \cdot (-2)^3 = 6, \\ P(-1) = 4 &\implies a_0 + a_1 \cdot (-1) + a_2 \cdot (-1)^2 + a_3 \cdot (-1)^3 = 4, \\ P(1) = 6 &\implies a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 + a_3 \cdot 1^3 = 6, \\ P(2) = 22 &\implies a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 = 22. \end{aligned}$$

Simplifying gives us

$$\begin{aligned} a_0 - 2a_1 + 4a_2 - 8a_3 &= 6 \\ a_0 - a_1 + a_2 - a_3 &= 4 \\ a_0 + a_1 + a_2 + a_3 &= 6 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 22. \end{aligned}$$

- b) The corresponding augmented matrix is

$$\left( \begin{array}{cccc|c} 1 & -2 & 4 & -8 & 6 \\ 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 8 & 22 \end{array} \right)$$

We label pivots with boxes as we proceed along. First, we subtract row 1 from each of rows 2, 3, and 4.

$$\left( \begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 8 & 22 \end{array} \right) \rightsquigarrow \left( \begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 3 & -3 & 9 & 0 \\ 0 & 4 & 0 & 16 & 16 \end{array} \right)$$

We now create zeros below the second pivot by subtracting multiples of the second row, then divide by 6 to get

$$\left( \begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{6} & -12 & 6 \\ 0 & 0 & 12 & -12 & 24 \end{array} \right) \xrightarrow{R_3 = R_3 \div 6} \left( \begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 12 & -12 & 24 \end{array} \right).$$

Now we subtract a 12 times row 3 from row 4 and divide by 12:

$$\left( \begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{12} & 12 \end{array} \right) \xrightarrow{R_4 = R_4 \div 12} \left( \begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{1} & \boxed{1} \end{array} \right).$$

At this point we can actually use back-substitution to solve: the last row says  $a_3 = 1$ , then plugging in  $a_3 = 1$  in the third row gives us  $a_2 = 3$ , etc. However, for the sake of practice with reduced echelon form, let's keep row-reducing.

From right to left, we create zeros above the pivots in pivot columns by subtracting multiples of the pivot columns.

$$\begin{aligned} \left( \begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right) & \xrightarrow{\quad} \left( \begin{array}{cccc|c} \boxed{1} & -2 & 4 & 0 & 14 \\ 0 & \boxed{1} & -3 & 0 & -9 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right) \\ & \xrightarrow{\quad} \left( \begin{array}{cccc|c} \boxed{1} & -2 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right) \\ & \xrightarrow{\quad} \left( \begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right) \end{aligned}$$

So  $a_0 = 2$ ,  $a_1 = 0$ ,  $a_2 = 3$ , and  $a_3 = 1$ . In other words,

$$\boxed{P(x) = 2 + 3x^2 + x^3}.$$

Therefore, there is exactly one third-degree polynomial satisfying the conditions of the problem. (You should check that, in fact, we have  $P(-2) = 6$ ,  $P(-1) = 4$ , etc.)