**Supplemental problems: §3.4**

**1.** Consider $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+y \\ x-y \end{pmatrix}$$

and $U\colon \mathbf{R}^3 \to \mathbf{R}^2$ defined by first projecting onto the $xy$-plane (forgetting the $z$-coordinate), then rotating counterclockwise by 90°.

**a)** Compute the standard matrices $A$ and $B$ for $T$ and $U$, respectively.

**b)** Compute the standard matrices for $T \circ U$ and $U \circ T$.

**c)** Circle all that apply:
$T \circ U$ is: one-to-one onto

$U \circ T$ is: one-to-one onto

**2.** Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation which projects onto the $yz$-plane and then forgets the $x$-coordinate, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by 60°. Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

**a)** Which composition makes sense? (Circle one.)

$$U \circ T \qquad T \circ U$$

**b)** Find the standard matrix for the transformation that you circled in (b).

**3.** Find all matrices $B$ that satisfy

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}.$$

**4.** Let $T$ and $U$ be the (linear) transformations below:

$T(x_1, x_2, x_3) = (x_3 - x_1,\ x_2 + 4x_3,\ x_1,\ 2x_2 + x_3) \qquad U(x_1, x_2, x_3, x_4) = (x_1 - 2x_2,\ x_1)$.

**a)** Which compositions makes sense (circle all that apply)? $U \circ T \qquad T \circ U$

**b)** Compute the standard matrix for $T$ and for $U$.

**c)** Compute the standard matrix for each composition that you circled in (a).

**5.** True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

**a)** If $A$ and $B$ are matrices and the products $AB$ and $BA$ are both defined, then $A$ and $B$ must be square matrices with the same number of rows and columns.

**b)** If $A$, $B$, and $C$ are nonzero $2 \times 2$ matrices satisfying $BA = CA$, then $B = C$.

**c)** Suppose $A$ is an $4 \times 3$ matrix whose associated transformation $T(x) = Ax$ is not one-to-one. Then there must be a $3 \times 3$ matrix $B$ which is not the zero matrix and satisfies $AB = 0$.

**d)** Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ and $U : \mathbf{R}^m \to \mathbf{R}^p$ are one-to-one linear transformations. Then $U \circ T$ is one-to-one. (What if $U$ and $T$ are not necessarily linear?)

**6.** In each case, use geometric intuition to either give an example of a matrix with the desired properties or explain why no such matrix exists.

**a)** A $3 \times 3$ matrix $P$, which is not the identity matrix or the zero matrix, and satisfies $P^2 = P$.

**b)** A $2 \times 2$ matrix $A$ satisfying $A^2 = I$.

**c)** A $2 \times 2$ matrix $A$ satisfying $A^3 = -I$.

## Supplemental problems: §3.5-3.6

**1.** **a)** Fill in: $A$ and $B$ are invertible $n \times n$ matrices, then the inverse of $AB$ is ________.

**b)** If the columns of an $n \times n$ matrix $Z$ are linearly independent, is $Z$ necessarily invertible? Justify your answer.

**c)** If $A$ and $B$ are $n \times n$ matrices and $ABx = 0$ has a unique solution, does $Ax = 0$ necessarily have a unique solution? Justify your answer.

**2.** Suppose $A$ is an invertible matrix and

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \qquad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \qquad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find $A$.