Math 1553 Supplement, §5.1 and §5.2

Solutions

These are additional practice problems after completing the worksheet.

1. Find a basis $\mathcal{B}$ for the (-1)-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Solution.

For $\lambda = -1$, we find $\mathrm{Nul}(Z - \lambda I)$.

$$\left(Z - \lambda I \mid 0 \right) = \left(Z + I \mid 0 \right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $x = -y$, $y = y$, and $z = 0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

A basis is $\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. We can check to ensure $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue -1:

$$Z \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+3 \\ -3+2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

2. Give an example of matrices A and B which have the same eigenvalues and the same algebraic multiplicities for each eigenvalue, but which are *not* similar. Justify why they are not similar.

Solution.

Many examples possible. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Both A and B have characteristic equation $\lambda^2 = 0$, so each has eigenvalue $\lambda = 0$ with algebraic multiplicity two. However, the only matrix similar to A is the zero matrix: if P is any invertible 2×2 matrix then $P^{-1}AP = P^{-1}0P = 0$. Therefore, A and B are not similar.

3. Using facts about determinants, justify the following fact: if A is an $n \times n$ matrix, then A and A^T have the same characteristic polynomial.

Solution.

We will use three facts which apply to all $n \times n$ matrices B, Y, Z:

(1) $\det(B) = \det(B^T)$.

(2) $(Y - Z)^T = Y^T - Z^T$

(3) If λ is any scalar then $(\lambda I)^T = \lambda I$ since the identity matrix is completely symmetric about its diagonal.

Using these three facts in order, we find

$$\det(A - \lambda I) = \det\big((A - \lambda I)^T\big) = \det\big(A^T - (\lambda I)^T\big) = \det(A^T - \lambda I).$$