Supplemental problems: §5.4

1. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A is an invertible matrix and A is diagonalizable, then A^{-1} is diagonalizable.

b) A diagonalizable $n \times n$ matrix admits n linearly independent eigenvectors.

c) If A is diagonalizable, then A has n distinct eigenvalues.

Solution.

a) True. If $A = PDP^{-1}$ and A is invertible then its eigenvalues are all nonzero, so the diagonal entries of D are nonzero and thus D is invertible (pivot in every diagonal position). Thus, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$.

b) True. By the Diagonalization Theorem, an $n \times n$ matrix is diagonalizable *if and only if* it admits n linearly independent eigenvectors.

c) False. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal but has only one eigenvalue.

2. Give examples of 2×2 matrices with the following properties. Justify your answers.

a) A matrix A which is invertible and diagonalizable.

b) A matrix B which is invertible but not diagonalizable.

c) A matrix C which is not invertible but is diagonalizable.

d) A matrix D which is neither invertible nor diagonalizable.

Solution.

a) We can take any diagonal matrix with nonzero diagonal entries:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) A shear has only one eigenvalue $\lambda = 1$. The associated eigenspace is the x-axis, so there do not exist two linearly independent eigenvectors. Hence it is not diagonalizable.

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

c) We can take any diagonal matrix with some zero diagonal entries:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

d) Such a matrix can only have the eigenvalue zero — otherwise it would have two eigenvalues, hence be diagonalizable. Thus the characteristic polynomial is $f(\lambda)=\lambda^2$. Here is a matrix with trace and determinant zero, whose zero-eigenspace (i.e., null space) is not all of $\mathbf{R}^2$:

$$D=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

3. $A=\begin{pmatrix}2&3&1\\3&2&4\\0&0&-1\end{pmatrix}.$

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A=CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0=\det(A-\lambda I)$.

$$0=\det\begin{pmatrix}2-\lambda&3&1\\3&2-\lambda&4\\0&0&-1-\lambda\end{pmatrix}=(-1-\lambda)(-1)^6\det\begin{pmatrix}2-\lambda&3\\3&2-\lambda\end{pmatrix}=(-1-\lambda)((2-\lambda)^2-9)$$

$$=(-1-\lambda)(\lambda^2-4\lambda-5)=-(\lambda+1)^2(\lambda-5).$$

So $\lambda=-1$ and $\lambda=5$ are the eigenvalues.

$\underline{\lambda=-1}$: $\left(A+I\mid 0\right)=\left(\begin{array}{ccc|c}3&3&1&0\\3&3&4&0\\0&0&0&0\end{array}\right)\xrightarrow{R_2=R_2-R_1}\left(\begin{array}{ccc|c}3&3&1&0\\0&0&1&0\\0&0&0&0\end{array}\right)\xrightarrow[\text{then } R_1=R_1/3]{R_1=R_1-R_2}$

$\left(\begin{array}{ccc|c}1&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right)$, with solution $x_1=-x_2$, $x_2=x_2$, $x_3=0$. The (-1)-eigenspace has basis $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$.

$\underline{\lambda=5}$:

$$\left(A-5I\mid 0\right)=\left(\begin{array}{ccc|c}-3&3&1&0\\3&-3&4&0\\0&0&-6&0\end{array}\right)\xrightarrow[R_3=R_3/(-6)]{R_2=R_2+R_1}\left(\begin{array}{ccc|c}-3&3&1&0\\0&0&5&0\\0&0&1&0\end{array}\right)\xrightarrow[\text{then } R_2\leftrightarrow R_3,\ R_1/(-3)]{R_1=R_1-R_3,\ R_2=R_2-5R_3}\left(\begin{array}{ccc|c}1&-1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right),$$

with solution $x_1=x_2,\ x_2=x_2,\ x_3=0$. The 5-eigenspace has basis $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$.

b) A is a 3×3 matrix that only admits 2 linearly independent eigenvectors, so A is not diagonalizable.

4. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.
The characteristic polynomial for A is $-\lambda^3 + 7\lambda^2 - 16\lambda + 12$, and $\lambda - 3$ is a factor. Decide if A is diagonalizable. If it is, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

Solution.

By polynomial division,

$$\frac{-\lambda^3 + 7\lambda^2 - 16\lambda + 12}{\lambda - 3} = -\lambda^2 + 4\lambda - 4 = -(\lambda - 2)^2.$$

Thus, the characteristic poly factors as $-(\lambda-3)(\lambda-2)^2$, so the eigenalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

For $\lambda_1 = 3$, we row-reduce $A - 3I$:

$$\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \xrightarrow[\text{(New } R_1)/3]{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix}$$

$$\xrightarrow[\text{then } R_2 = -R_2]{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - 6R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solutions to $\left(A - 3I \mid 0\right)$ are $x_1 = 2x_3,\ x_2 = -2x_3,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}. \qquad \text{The 3-eigenspace has basis } \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda_2 = 2$, we row-reduce $A - 2I$:

$$\begin{pmatrix} 6 & 36 & 62 \\ -6 & -36 & -62 \\ 3 & 18 & 31 \end{pmatrix} \quad \overset{\text{rref}}{\rightsquigarrow} \quad \begin{pmatrix} 1 & 6 & \frac{31}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solutions to $\left(A - 2I \quad 0\right)$ are $x_1 = -6x_2 - \frac{31}{3}x_3,\ x_2 = x_2,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6x_2 - \frac{31}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The 2-eigenspace has basis $\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore, $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 2 & -6 & -\frac{31}{3} \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that we arranged the eigenvectors in C in order of the eigenvalues 3, 2, 2, so we had to put the diagonals of D in the same order.

5. Which of the following 3×3 matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A matrix with one real eigenvalue.
3. A matrix with a real eigenvalue λ of algebraic multiplicity 2, such that the λ-eigenspace has dimension 2.
4. A matrix with a real eigenvalue λ such that the λ-eigenspace has dimension 2.

Solution.

The matrices in 1 and 3 are diagonalizable. A matrix with three distinct real eigenvalues automatically admits three linearly independent eigenvectors. If a matrix A has a real eigenvalue λ_1 of algebraic multiplicity 2, then it has another real eigenvalue λ_2 of algebraic multiplicity 1. The two eigenspaces provide three linearly independent eigenvectors.

The matrices in 2 and 4 need not be diagonalizable.

6. Suppose a 2×2 matrix A has eigenvalue $\lambda_1 = -2$ with eigenvector $v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$, and eigenvalue $\lambda_2 = -1$ with eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a) Find A.

b) Find A^{100}.

Solution.

a) We have $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

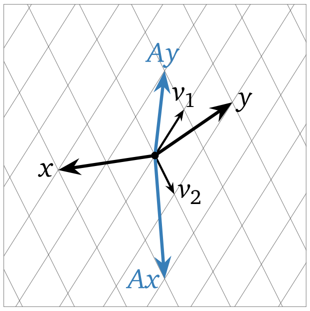
We compute $C^{-1} = \dfrac{1}{-5/2}\begin{pmatrix} -1 & -1 \\ -1 & 3/2 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$.

$$A = CDC^{-1} = \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -8 & -3 \\ -2 & -7 \end{pmatrix}.$$

b)

$$\begin{aligned}A^{100} = CD^{100}C^{-1} &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \cdot D^{100} \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2\cdot 2^{100} & 2\cdot 2^{100} \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3\cdot 2^{100}+2 & 3\cdot 2^{100}-3 \\ 2^{101}-2 & 2^{101}+3 \end{pmatrix}.\end{aligned}$$

7. Suppose that $A = C\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}C^{-1}$, where C has columns v_1 and v_2. Given x and y in the picture below, draw the vectors Ax and Ay.



Solution.

A does the same thing as $D = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}$, but in the v_1, v_2-coordinate system. Since D scales the first coordinate by $1/2$ and the second coordinate by -1, hence A scales the v_1-coordinate by $1/2$ and the v_2-coordinate by -1.

Supplemental problems: §5.5

1. **a)** If A is the matrix that implements rotation by 143° in $\mathbf{R}^2$, then A has no real eigenvalues.

b) A 3×3 matrix can have eigenvalues $3, 5$, and $2+i$.

c) If $v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$, then $w = \begin{pmatrix} 2i-1 \\ i \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$.

Solution.

a) True. If A had a real eigenvalue λ, then we would have $Ax = \lambda x$ for some nonzero vector x in $\mathbf{R}^2$. This means that x would lie on the same line through the origin as the rotation of x by 143°, which is impossible.

b) False. If $2+i$ is an eigenvalue then so is its conjugate $2-i$.

c) True. Any nonzero complex multiple of v is also an eigenvector for eigenvalue $1-i$, and $w = iv$.

2. Consider the matrix

$$A = \begin{pmatrix} 3\sqrt{3}-1 & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1 \end{pmatrix}$$

a) Find both complex eigenvalues of A.

b) Find an eigenvector corresponding to each eigenvalue.

Solution.

a) We compute the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\begin{pmatrix} 3\sqrt{3}-1-\lambda & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1-\lambda \end{pmatrix} \\ &= (-1-\lambda+3\sqrt{3})(-1-\lambda-3\sqrt{3}) + (2)(5)(3) \\ &= (-1-\lambda)^2 - 9(3) + 10(3) \\ &= \lambda^2 + 2\lambda + 4. \end{aligned}$$

By the quadratic formula,

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(4)}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i.$$

b) Let $\lambda = -1 - \sqrt{3}i$. Then

$$A - \lambda I = \begin{pmatrix} (i+3)\sqrt{3} & -5\sqrt{3} \\ 2\sqrt{3} & (i-3)\sqrt{3} \end{pmatrix}.$$

Since $\det(A - \lambda I) = 0$, the second row is a multiple of the first, so a row echelon form of A is

$$\begin{pmatrix} i+3 & -5 \\ 0 & 0 \end{pmatrix}.$$

Hence an eigenvector with eigenvalue $-1-\sqrt{3}i$ is $v = \begin{pmatrix} 5 \\ 3+i \end{pmatrix}$. It follows that an eigenvector with eigenvalue $-1+\sqrt{3}i$ is $\overline{v} = \begin{pmatrix} 5 \\ 3-i \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue of A, find a corresponding eigenvector.

Solution.

First we compute the characteristic polynomial by expanding cofactors along the third row:

$$\begin{aligned} f(\lambda) = \det\begin{pmatrix} 4-\lambda & -3 & 3 \\ 3 & 4-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix} &= (2-\lambda)\det\begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix} \\ &= (2-\lambda)\big((4-\lambda)^2+9\big) = (2-\lambda)(\lambda^2-8\lambda+25). \end{aligned}$$

Using the quadratic equation on the second factor, we find the eigenvalues

$$\lambda_1 = 2 \qquad \lambda_2 = 4-3i \qquad \overline{\lambda}_2 = 4+3i.$$

Next compute an eigenvector with eigenvalue $\lambda_1 = 2$:

$$A - 2I = \begin{pmatrix} 2 & -3 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = 0$, $y = z$, so the parametric vector form of the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \overset{\text{eigenvector}}{\rightsquigarrow} v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Now we compute an eigenvector with eigenvalue $\lambda_2 = 4-3i$:

$$A = (4-3i)I = \begin{pmatrix} 3i & -3 & 3 \\ 3 & 3i & -2 \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 3 & 3i & -2 \\ 3i & -3 & 3 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - iR_1} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 3+2i \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_2 = R_2 \div (3+2i)} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{\text{row replacements}} \begin{pmatrix} 3 & 3i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 \div 3} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form of the solution is $x = -iy$, $z = 0$, so the parametric vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{eigenvector}} v_2 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

An eigenvector for the complex conjugate eigenvalue $\overline{\lambda}_2 = 4 + 3i$ is the complex conjugate eigenvector $\overline{v}_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.