Math 1553 Supplement, §5.6 and §6.1

Solutions

1. Suppose p and q are real numbers on the open interval $(0,1)$, and

$$A = \begin{pmatrix} p & 1-q \\ 1-p & q \end{pmatrix}$$

(1) Is A stochastic? Why?
(2) Does A have unique steady state vector? Why?
(3) By inspection(without computation), give an eigenvalue of A.
(4) Compute the steady-state vector of A.
(5) Compute the limit

$$\lim_{n\to\infty} A^n$$

Solution.

(1) Yes: columns sum to 1, elements strictly positive
(2) Yes: all elements positive so Perron–Frobenius theorem applys.
(3) $\lambda = 1$
(4) solve $(A-I)v = v$ and find a probability vector from the solution we see

$$v = \frac{1}{2-p-q}\begin{pmatrix} 1-q \\ 1-p \end{pmatrix}$$

(5) Denote the limit matrix as B. What is Be_1, Be_2? (recall yourself Perron-Frobenius theorem) The limiting matrix will be

$$B = \frac{1}{2-p-q}\begin{pmatrix} 1-q & 1-q \\ 1-p & 1-p \end{pmatrix}$$

2. $y = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad (u)_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

(1) Determine whether u_1 and u_2
   (a) are linearly independent
   (b) are mutually orthogonal
   (c) are orthonormal
   (d) span $\mathbf{R}^3$
(2) Is y in $W = \text{Span}\{u_1, u_2\}$?
(3) Compute the vector, $\widehat{y} \in W$, that most closely approximates y. [You may need orthogonal projections from §6.2]
(4) Construct a vector, z, that is in $W^\perp$.
(5) Make a rough sketch(use online tools) of $u_1, u_2, y, \widehat{y}$, and z.

Solution.

(1) By inspection, the vectors u_1 and u_2 are linearly independent. Computing $u_1 \cdot u_2 = 0$ implies they are also mutually orthogonal. Therefore they span $\mathbb{R}^3$.
(2) By inspection, y is not in the span because it has a non-zero x_3 component.

(3) The projection of y onto W is calculated in the usual way, with

$$\widehat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

(4) $z = y - \widehat{y}$