Math 1553 Worksheet §5.1 and §5.2

Solutions

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. In every case, assume that A is an $n \times n$ matrix.

a) If v_1 and v_2 are linearly independent eigenvectors of A, then they must correspond to different eigenvalues.

b) The entries on the main diagonal of A are the eigenvalues of A.

c) The number λ is an eigenvalue of A if and only if there is a nonzero solution to the equation $(A-\lambda I)x = 0$.

d) To find the eigenvectors of A, we reduce the matrix A to row echelon form.

e) If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

Solution.

a) False. For example, if $A = I_2$ then e_1 and e_2 are linearly independent eigenvectors both corresponding to the eigenvalue $\lambda = 1$.

b) False. This is true if A is triangular, but not in general.

For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda)-(1)(-1)=0 \qquad \lambda^2-2\lambda+1=0 \qquad (\lambda-1)^2=0 \qquad \lambda=1.$$

c) True.

$$(A-\lambda I)x = 0 \iff Ax - \lambda x = 0 \iff Ax = \lambda x.$$

Therefore, $(A-\lambda I)x = 0$ has a nonzero solution if and only if $Ax = \lambda x$ has a nonzero solution, which is to say that λ is an eigenvalue of A.

d) False. The RREF of A will only compute the eigenvectors with eigenvalue zero, or will tell us that zero is not an eigenvalue. To get the eigenvectors corresponding to an eigenvalue λ, we put $A-\lambda I$ into RREF and write the solutions of $\left(A-\lambda I \mid 0\right)$ in parametric vector form.

e) True. Let v be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, v is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{2}$.

2. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

3. Let $A = \begin{pmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{pmatrix}$. Find the eigenvalues of A.

Solution.

We find the characteristic polynomial $\det(A - \lambda I)$ any way we like. The computation below first expands cofactors along the second row:

$$\begin{aligned} \det(A - \lambda I_3) = \det \begin{pmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{pmatrix} &= (1-\lambda)\det \begin{pmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{pmatrix} \\ &= (1-\lambda)\big((5-\lambda)(-2-\lambda) - 18\big) = (1-\lambda)\big(\lambda^2 - 3\lambda - 28\big) \\ &= (1-\lambda)(\lambda-7)(\lambda+4) \end{aligned}$$

The characteristic equation is thus $(1-\lambda)(\lambda-7)(\lambda+4) = 0$, so the eigenvalues are $\lambda = -4$, $\lambda = 1$, and $\lambda = 7$.