Math 1553
Introduction to Linear Algebra

School of Mathematics
Georgia Institute of Technology

Chapter 0

Overview

Linear. Algebra.

What is Linear Algebra?

Linear

- having to do with lines/planes/etc.
- For example, $x + y + 3z = 7$, not $\sin, \log, x^2$, etc.

Algebra

- solving equations involving numbers and symbols
- from al-jebr (Arabic), meaning reunion of broken parts
- 9th century Abu Ja'far Muhammad ibn Muso al-Khwarizmi

Why a whole course?

But these are the easiest kind of equations! I learned how to solve them in 7th grade!

Ah, but engineers need to solve *lots* of equations in *lots* of variables.

$$\begin{aligned}
3x_1 + 4x_2 + 10x_3 + 19x_4 - 2x_5 - 3x_6 &= 141 \\
7x_1 + 2x_2 - 13x_3 - 7x_4 + 21x_5 + 8x_6 &= 2567 \\
-x_1 + 9x_2 + \tfrac{3}{2}x_3 + x_4 + 14x_5 + 27x_6 &= 26 \\
\tfrac{1}{2}x_1 + 4x_2 + 10x_3 + 11x_4 + 2x_5 + x_6 &= -15
\end{aligned}$$

Often, it's enough to know some information about the set of solutions without having to solve the equations at all!

Also, what if one of the coefficients of the x_i is itself a parameter— like an unknown real number t?

In real life, the difficult part is often in recognizing that a problem can be solved using linear algebra in the first place: need *conceptual* understanding.

Large classes of engineering problems, no matter how huge, can be reduced to linear algebra:

$$Ax = b \quad \text{or}$$
$$Ax = \lambda x$$

"...and now it's just linear algebra"

Applications of Linear Algebra

Civil Engineering: How much traffic flows through the four labeled segments?

Traffic flow (cars/hr)
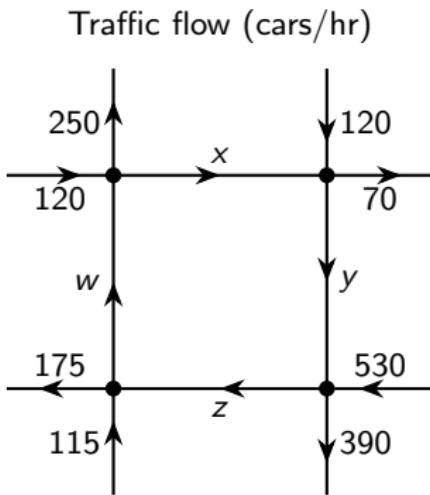
⤳ system of linear equations:
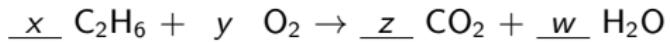
$$w + 120 = x + 250$$
$$x + 120 = y + 70$$
$$y + 530 = z + 390$$
$$z + 115 = w + 175$$

Chemistry: Balancing reaction equations

$$\underline{x}\ C_2H_6 + \underline{y}\ O_2 \rightarrow \underline{z}\ CO_2 + \underline{w}\ H_2O$$

⤳ system of linear equations, one equation for each element.

$$2x = z$$
$$6x = 2w$$
$$2y = 2z + w$$

Applications of Linear Algebra

Biology: In a population of rabbits. . .

- ▶ half of the new born rabbits survive their first year
- ▶ of those, half survive their second year
- ▶ the maximum life span is three years
- ▶ rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

⤳ system of linear equations:

$$
\begin{aligned}
6y_{2016} + 8z_{2016} &= x_{2017} \\
\tfrac{1}{2}x_{2016} \quad\quad &= y_{2017} \\
\tfrac{1}{2}y_{2016} \quad\quad &= z_{2017}
\end{aligned}
$$

Question

Does the rabbit population have an asymptotic behavior? Is this even a linear algebra question? Yes, it is!   [interactive]

Geometry and Astronomy: Find the equation of a circle passing through 3 given points, say $(1, 0)$, $(0, 1)$, and $(1, 1)$. The general form of a circle is $a(x^2 + y^2) + bx + cy + d = 0$.

⤳ system of linear equations:

$$
\begin{aligned}
a + b \qquad + d &= 0 \\
a \qquad + c + d &= 0 \\
2a + b + c + d &= 0
\end{aligned}
$$

Very similar to: compute the orbit of a planet:

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

Applications of Linear Algebra

Google: "The 25 billion dollar eigenvector." Each web page has some importance, which it shares via outgoing links to other pages ⤳ system of linear equations (in gazillions of variables).

Larry Page flies around in a private 747 because he paid attention in his linear algebra class!

- Solve the matrix equation $Ax = b$
  - Solve systems of linear equations using matrices, row reduction, and inverses.
  - Solve systems of linear equations with varying parameters using parametric forms for solutions, the geometry of linear transformations, the characterizations of invertible matrices, and determinants.

- Solve the matrix equation $Ax = \lambda x$
  - Solve eigenvalue problems through the use of the characteristic polynomial.
  - Understand the dynamics of a linear transformation via the computation of eigenvalues, eigenvectors, and diagonalization.

- Almost solve the equation $Ax = b$
  - Find best-fit solutions to systems of linear equations that have no actual solution using least squares approximations.

What to Expect This Semester

Your previous math courses probably focused on how to do (sometimes rather involved) computations.

- Compute the derivative of $\sin(\log x)\cos(e^x)$.
- Compute $\int_0^1 (1 - \cos(x))\, dx$.

This is important, **but** Matlab can do all these problems better than any of us can. Nobody is going to hire you to do something a computer can do better.

> If a computer can do the problem better than you can, then it's just an algorithm: **this is not problem solving**.

So what are we going to do?

- About half the material focuses on how to do linear algebra computations—that is still important.
- The other half is on *conceptual* understanding of linear algebra. This is much more subtle: it's about figuring out *what question* to ask the computer, or whether you actually need to do any computations at all.

Interactive Linear Algebra

Dan Margalit and Joe Rabinoff have written a free online textbook called *Interactive Linear Algebra*, with a version specifically created for this course.

`https://textbooks.math.gatech.edu/ila/1553/`

The content of the course (i.e., the material that is fair game for exams) is exactly what you see in the textbook. This does not include the parts you have to click to uncover, which are generally examples and remarks.

There are about 150 interactive demonstrations in the book. They're there for a reason: you'll be expected to gain and demonstrate a **geometric** understanding of the material.

▶ **Practice, practice, practice!** It makes sense to most people that if you want to get good at tennis, you have to hit a million forehands and backhands. But for some reason, many people think you're either born good at math, or you're not. This is ridiculous. People who are good at math are just people who have spent a long time thinking about math. *Nobody* is born good at math.

Not good at math ⟶

▶ **Do the homework carefully.** Homework is practice for the quizzes. Quizzes are practice for the midterms. Remember what I said about practice?

▶ **Study the pictures.** I expect you to play around with the demos in the book until you understand them!

▶ **Take advantage of the resources provided.** Come to office hours! Read the textbook! Go to Math Lab!

Course Administration

- Homework is on WeBWorK (access through Canvas), and is due *Thursdays at 11:59pm* (except the Warmup which does not count and is listed as due Friday).

- Quizzes happen in studio most weeks.

- Piazza polls measure class participation. Sign up for Piazza through Canvas, *with your Canvas email address*. It's easiest if you then download the Piazza app on your phone.

- Exams: there are three midterms, and a cumulative final.

Resources

On the webpage you'll find:

- ▶ The textbook: Interactive Linear Algebra is online-only.
- ▶ Course materials: practice exams, worksheet solutions, etc.
- ▶ Course organization: grading policies, details about homework and exams, etc.
- ▶ Help and advice: how to succeed in this course, resources available to you.
- ▶ Calendar: what will happen on which day, when the midterms are, etc.

Canvas: your grades, links to Piazza and WeBWorK, announcements.

Piazza: this is where to ask questions.

WeBWorK: our online homework system.

Chapter 1

Systems of Linear Equations: Algebra

Section 1.1

Systems of Linear Equations

Line, Plane, Space, . . .

Recall that **R** denotes the collection of all real numbers, i.e. the number line. It contains numbers like $0, -1, \pi, \frac{3}{2}, \ldots$
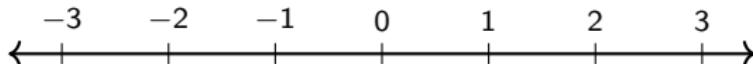
Definition

Let n be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

Example

When $n = 1$, we just get **R** back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the *number line*.
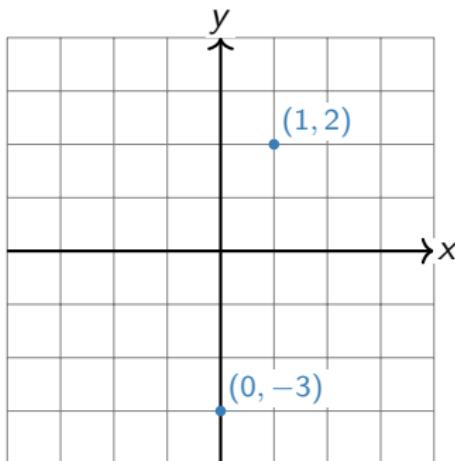
Example

When $n = 2$, we can think of $\mathbf{R}^2$ as the *plane*. This is because every point on the plane can be represented by an ordered pair of real numbers, namely, its x- and y-coordinates.
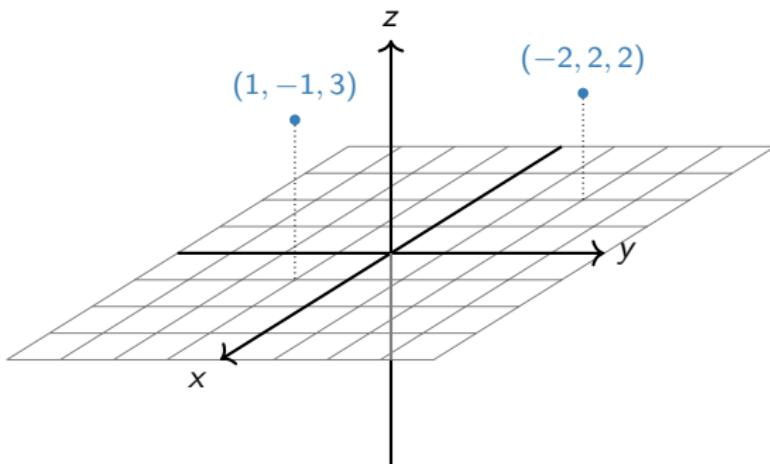
Example

When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. This is because every point in space can be represented by an ordered triple of real numbers, namely, its x-, y-, and z-coordinates.

So what is $\mathbf{R}^4$? or $\mathbf{R}^5$? or $\mathbf{R}^n$?

. . . go back to the *definition*: ordered n-tuples of real numbers

$$(x_1, x_2, x_3, \ldots, x_n).$$

They're still "geometric" spaces, in the sense that our intuition for $\mathbf{R}^2$ and $\mathbf{R}^3$ sometimes extends to $\mathbf{R}^n$, but they're harder to visualize.
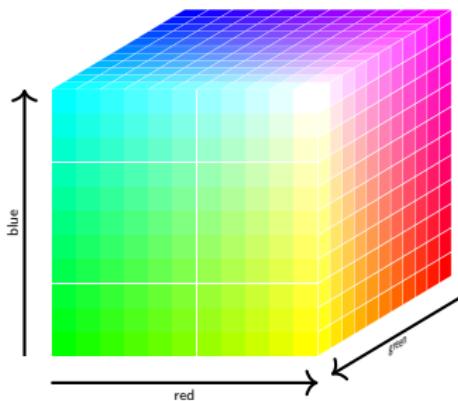
We'll make definitions and state theorems that apply to any $\mathbf{R}^n$, but we'll only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

The power of using these spaces is the ability to use elements of $\mathbf{R}^n$ to *label* various objects of interest, like solutions to systems of equations.
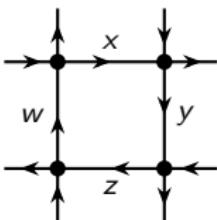
All colors you can see can be described by three quantities: the amount of red, green, and blue light in that color. Therefore, we can use the elements of $\mathbf{R}^3$ to *label* all colors: the point (.2, .4, .9) labels the color with 20% red, 40% green, and 90% blue.

Last time we could have used $\mathbf{R}^4$ to *label* the amount of traffic (x, y, z, w) passing through four streets.
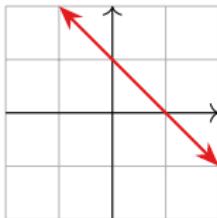

For instance the point $(100, 20, 30, 150)$ corresponds to a situation where 100 cars per hour drive on road x, 20 cars per hour drive on road y, etc.

One Linear Equation

What does the solution set of a linear equation look like?
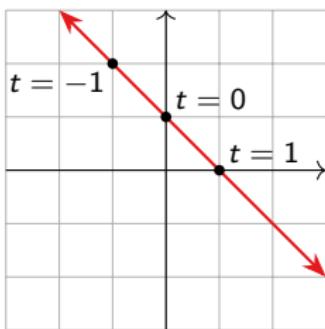
$x + y = 1$ ⤳ a line in the plane: $y = 1 - x$
This is called the **implicit equation** of the line.

We can write the same line in **parametric form** in $\mathbf{R}^2$:

$$(x, y) = (t, 1 - t) \qquad t \text{ in } \mathbf{R}.$$

This means that every point on the line has the form $(t, 1 - t)$ for some real number t.   Note we are using $\mathbf{R}$ to *label* the points on a line in $\mathbf{R}^2$.

Aside
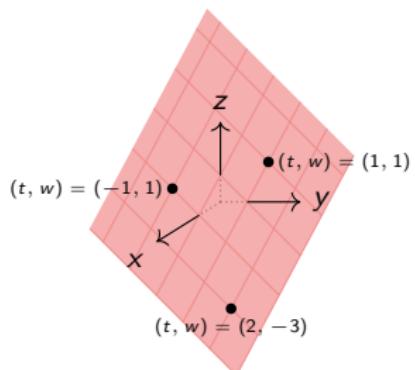
What is a line? A ray that is *straight* and infinite in both directions.

What does the solution set of a linear equation look like?

$x + y + z = 1$ ⤳ a plane in space:
This is the **implicit equation** of the plane.

[interactive]

Does this plane have a **parametric form**?

$$(x, y, z) = (1 - t - w, \ t, \ w) \qquad t, w \text{ in } \mathbf{R}.$$

Note we are using $\mathbf{R}^2$ to *label* the points on a plane in $\mathbf{R}^3$.

Aside
What is a plane? A flat sheet of paper that's infinite in all directions.

What does the solution set of a linear equation look like?

$x + y + z + w = 1$ ⤳ a "3-plane" in "4-space"...   [not pictured here]
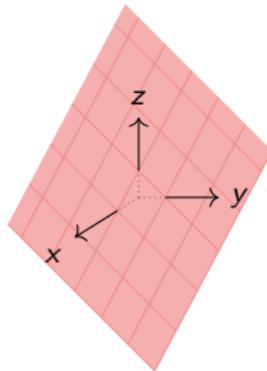
Poll

Everybody get out your gadgets!

Poll

Is the plane from the previous example equal to $\mathbf{R}^2$?

A. Yes   **B.** No

No! Every point on this plane is in $\mathbf{R}^3$: that means it has three coordinates. For instance, $(1, 0, 0)$. Every point in $\mathbf{R}^2$ has two coordinates. But we can *label* the points on the plane by $\mathbf{R}^2$.
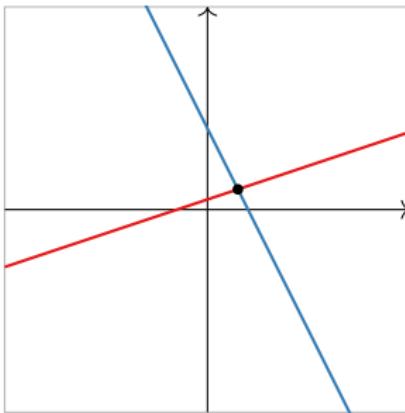
What does the solution set of a *system* of more than one linear equation look like?

$$x - 3y = -3$$
$$2x + y = 8$$

... is the *intersection* of two lines, which is a *point* in this case.

In general it's an intersection of lines, planes, etc.

[two planes intersecting]

In what other ways can two lines intersect?

$$x - 3y = -3$$
$$x - 3y = 3$$

has no solution: the lines are
parallel.

A system of equations with no solutions is called **inconsistent**.
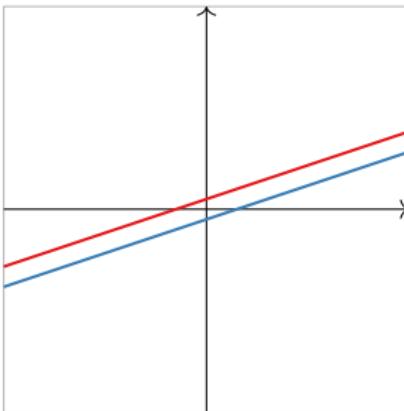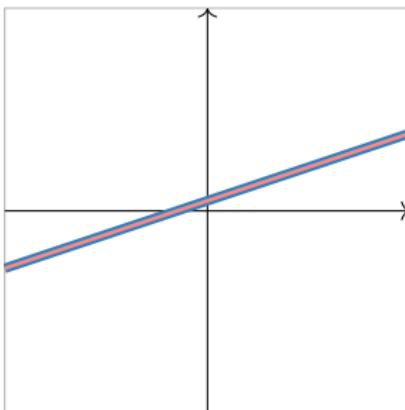
Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$
$$2x - 6y = -6$$

has infinitely many solutions:
they are the *same line*.

Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Summary

- $\mathbf{R}^n$ is the set of ordered lists of n numbers.
- $\mathbf{R}^n$ can be used to label geometric objects, like $\mathbf{R}^2$ can label points on a plane.
- The solutions of a system equations look like an intersection of lines, planes, etc.
- Finding all the solutions of a system of equations means finding a **parametric form**: a labeling by some $\mathbf{R}^n$.

Solving Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

This is the kind of problem we'll talk about for the first half of the course.

> - A **solution** is a list of numbers $x, y, z, \ldots$ that makes *all* of the equations true.
> - The **solution set** is the collection of all solutions.
> - **Solving** the system means finding the solution set in a "parameterized" form.

What is a *systematic* way to solve a system of equations?

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

What strategies do you know?

- ▶ Substitution
- ▶ Elimination

Both are perfectly valid, but only elimination scales well to large numbers of equations.

Example

Solve the system of equations

$$\begin{aligned}
x + 2y + 3z &= 6 \\
2x - 3y + 2z &= 14 \\
3x + y - z &= -2
\end{aligned}$$

Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. **(scale)**
- Add a multiple of one equation to another. **(replacement)**
- Swap two equations. **(swap)**

It sure is a pain to have to write x, y, z, and $=$ over and over again.

Matrix notation: write just the numbers, in a box, instead!

$$
\begin{array}{rl}
x + 2y + 3z &= 6 \\
2x - 3y + 2z &= 14 \\
3x + y - z &= -2
\end{array}
\quad \text{becomes} \quad
\left(\begin{array}{ccc|c}
1 & 2 & 3 & 6 \\
2 & -3 & 2 & 14 \\
3 & 1 & -1 & -2
\end{array}\right)
$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

▶ Multiply all entries in a row by a nonzero number.   **(scale)**
▶ Add a multiple of each entry of one row to the corresponding entry in another.   **(row replacement)**
▶ Swap two rows.   **(swap)**

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Start:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{aligned} x \quad\quad &= A \\ y \quad &= B \\ z &= C \end{aligned} \qquad \text{or in matrix form,} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)$$

So we need to do row operations that make the start matrix look like the end one.

Strategy (preliminary): fiddle with it so we only have ones and zeros. [animated]

Row Operations

Continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$R_2 = R_2 - 2R_1$

$R_3 = R_3 - 3R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that
would produce ugly fractions.

Let's swap the last two rows first.

$R_2 \longleftrightarrow R_3$

$R_2 = R_2 \div -5$

$R_3 = R_3 + 7R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & \boxed{3} & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 \div 10}$$

$$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

We want these to be zero.

Let's make this a 1 first.

$$\xrightarrow{R_2 = R_2 - 2R_3, \ R_1 = R_1 - 3R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - 2R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{\text{translates into}}$$

$$\begin{aligned} x & & & = & 1 \\ & y & & = & -2 \\ & & z & = & 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

$$\xrightarrow{\text{substitute solution}}$$

$$\begin{aligned} 1 + 2 \cdot (-2) + 3 \cdot 3 &= 6 \\ 2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 &= 14 \\ 3 \cdot 1 + (-2) - 3 &= -2 \end{aligned}$$

✔

Row Equivalence

> **Important**
>
> The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition
Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

A Bad Example

Example

Solve the system of equations

$$x + y = 2$$
$$3x + 4y = 5$$
$$4x + 5y = 9$$

Let's try doing row operations: [interactive row reducer]

First clear these by subtracting multiples of the first row. $\longrightarrow$

$$\begin{pmatrix} 1 & 1 & | & 2 \\ 3 & 4 & | & 5 \\ 4 & 5 & | & 9 \end{pmatrix} \xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 4 & 5 & | & 9 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 4R_1} \begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 0 & 1 & | & 1 \end{pmatrix}$$

Now clear this by subtracting the second row. $\longrightarrow$

$$\begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 0 & 1 & | & 1 \end{pmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 0 & 0 & | & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & -1 \\ 0 & 0 & | & 2 \end{pmatrix} \quad \text{translates into} \xrightarrow{} \quad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

Section 1.2

Row Reduction

Row Echelon Form

Let's come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. What do we mean by "solved"?

> A matrix is in **row echelon form** if
> 1. All zero rows are at the bottom.
> 2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
> 3. Below a leading entry of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$ = any number

$\boxed{\star}$ = any nonzero number

Definition
A **pivot** $\boxed{\star}$ is the first nonzero entry of a row of a matrix. A **pivot column** is a column containing a pivot of a matrix *in row echelon form*.

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Picture:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star$ = any number
1 = pivot

Note: Echelon forms do not care whether or not a column is augmented. Just ignore the vertical line.

Question

Can every matrix be put into reduced row echelon form only using row operations?

Answer: Yes! Stay tuned.

Why is this the "solved" version of the matrix?

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

is in reduced row echelon form. It translates into

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3, \end{aligned}$$

which is clearly the solution.

But what happens if there are fewer pivots than rows?

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

... parametrized solution set (later).

Poll

Poll

Which of the following matrices are in reduced row echelon form?

A. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ E. $\begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$

F. $\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array} \right)$

Answer: B, D, E, F.

Note that A is in row echelon form though.

Summary

▶ Solving a system of equations means producing all values for the unknowns that make all the equations true simultaneously.

▶ It is easier to solve a system of linear equations if you put all the coefficients in an **augmented matrix**.

▶ Solving a system using the elimination method means doing **elementary row operations** on an augmented matrix.

▶ Two systems or matrices are **row-equivalent** if one can be obtained from the other by doing a sequence of elementary row operations. Row-equivalent systems have the *same solution set*.

▶ A linear system with no solutions is called **inconsistent**.

▶ The (reduced) row echelon form of a matrix is its "solved" row-equivalent version.

A matrix is in **row echelon form** if

1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
3. Below a leading entry of a row, all entries are *zero*.

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Row echelon form:

$$\begin{pmatrix} \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boxed{\star}$ = **pivots**

Row Reduction: Theorem

Theorem
Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm, called **row reduction** or **Gaussian elimination**, which demonstrates that every matrix is row equivalent to *at least one* matrix in reduced row echelon form.

Note: Like echelon forms, the row reduction algorithm does not care if a column is augmented: ignore the vertical line when row reducing.

The uniqueness statement is interesting—it means that, nomatter *how* you row reduce, you *always* get the same matrix in reduced row echelon form. (Assuming you only do the three legal row operations.) (And you don't make any arithmetic errors.)

Maybe you can figure out why it's true!

Row Reduction Algorithm

Step 1a Swap the 1st row with a lower one so a leftmost nonzero entry is in 1st row (if necessary).

Step 1b Scale 1st row so that its leading entry is equal to 1.

Step 1c Use row replacement so all entries below this 1 are 0.

Step 2a Swap the 2nd row with a lower one so that the leftmost nonzero entry is in 2nd row.

Step 2b Scale 2nd row so that its leading entry is equal to 1.

Step 2c Use row replacement so all entries below this 1 are 0.

Step 3a Swap the 3rd row with a lower one so that the leftmost nonzero entry is in 3rd row.

etc.

Last Step Use row replacement to clear all entries above the pivots, starting with the last pivot (to make life easier).

Example

$$\left(\begin{array}{rrr|r} 0 & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

[animated]

Row Reduction

Example

$$\left(\begin{array}{ccc|c} \boxed{0} & -7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Step 1a: Row swap to make this nonzero.

$$\xrightarrow{R_1 \longleftrightarrow R_2}$$

$$\left(\begin{array}{ccc|c} \boxed{2} & 4 & 6 & 12 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Step 1b: Scale to make this 1.

$$\xrightarrow{R_1 = R_1 \div 2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ \boxed{3} & 1 & -1 & -2 \end{array}\right)$$

Step 1c: Subtract a multiple of the first row to clear this.

$$\xrightarrow{R_3 = R_3 - 3R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

Optional: swap rows 2 and 3 to make Step 2b easier later on.

$$\xrightarrow{R_2 \longleftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Row Reduction
Example, continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Step 2a: This is already nonzero.
Step 2b: Scale to make this 1.

  (There are no fractions because
  of the optional step before.)

$\xrightarrow{R_2 = R_2 \div -5}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

Step 2c: Add 7 times
the second row to clear this.

$\xrightarrow{R_3 = R_3 + 7R_2}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

Note: Step 2 never messes up the first (nonzero) column of the matrix, because it looks like this:

"Active" row ⟶

$$\left(\begin{array}{ccc|c} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{array}\right)$$

$$\begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & \boxed{10} & | & 30 \end{pmatrix}$$

Step 3a: This is already nonzero.
Step 3b: Scale to make this 1.

$$R_3 = R_3 \div 10 \\ \rightsquigarrow$$

$$\begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

Note: Step 3 never messes up the columns to the left.
Note: The matrix is now in row echelon form!

$$\begin{pmatrix} 1 & 2 & \boxed{3} & | & 6 \\ 0 & 1 & \boxed{2} & | & 4 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

Last step: Add multiples of
the third row to clear these.

$$R_2 = R_2 - 2R_3 \\ \rightsquigarrow$$

$$R_1 = R_1 - 3R_3 \\ \rightsquigarrow$$

$$\begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \boxed{2} & 0 & | & -3 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

Last step: Add -2 times
the third row to clear this.

$$R_1 = R_1 - 2R_2 \\ \rightsquigarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}$$

Success! The reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array} \right) \quad \Longrightarrow \quad \left\{ \begin{array}{rcrcr} x & & & = & 1 \\ & y & & = & -2 \\ & & z & = & 3 \end{array} \right.$$

Recap

Get a 1 here

$$\begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} 1 & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix}$$

Get a 1 here

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & \star & \star & \star \\ 0 & \star & \star & \star \end{pmatrix}$$

(maybe these are already zero)

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Get a 1 here

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Clear down

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \star \end{pmatrix}$$

Matrix is in REF

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up

$$\begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Clear up

$$\begin{pmatrix} 1 & \star & \star & 0 \\ 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix is in RREF

$$\begin{pmatrix} 1 & 0 & \star & 0 \\ 0 & 1 & \star & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Profit?

The linear system

$$2x + 10y = -1$$
$$3x + 15y = 2$$

gives rise to the matrix

$$\left(\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array} \right).$$

Let's row reduce it: [interactive row reducer]

$$\left(\begin{array}{cc|c} 2 & 10 & -1 \\ 3 & 15 & 2 \end{array} \right)$$

$\xrightarrow{R_1 = R_1 \div 2}$

$$\left(\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 3 & 15 & 2 \end{array} \right) \qquad \text{(Step 1b)}$$

$\xrightarrow{R_2 = R_2 - 3R_1}$

$$\left(\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{array} \right) \qquad \text{(Step 1c)}$$

$\xrightarrow{R_2 = R_2 \times \frac{2}{7}}$

$$\left(\begin{array}{cc|c} 1 & 5 & -\frac{1}{2} \\ 0 & 0 & 1 \end{array} \right) \qquad \text{(Step 2b)}$$

$\xrightarrow{R_1 = R_1 + \frac{1}{2}R_2}$

$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array} \right) \qquad \text{(Step 2c)}$$

The row reduced matrix

$$\left(\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 1 \end{array} \right)$$

corresponds to the *inconsistent* system

$$x + 5y = 0$$
$$0 = 1.$$

Question

What does an augmented matrix in reduced row echelon form look like, if its system of linear equations is inconsistent?

Answer:

$$\left(\begin{array}{cccc|c} 1 & 0 & \star & \star & 0 \\ 0 & 1 & \star & \star & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

An augmented matrix corresponds to an inconsistent system of equations if and only if *the last* (i.e., the augmented) *column is a pivot column*.

Section 1.3

Parametric Form

Another Example

The linear system
$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned}$$
gives rise to the matrix
$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right).$$

Let's row reduce it: [interactive row reducer]

$$\left(\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right)$$

$\xrightarrow{\quad R_1 \longleftrightarrow R_2 \quad}$
$$\left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 2 & 1 & 12 & 1 \end{array}\right)$$
(Optional)

$\xrightarrow{\quad R_2 = R_2 - 2R_1 \quad}$
$$\left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & -3 & -6 & 3 \end{array}\right)$$
(Step 1c)

$\xrightarrow{\quad R_2 = R_2 \div -3 \quad}$
$$\left(\begin{array}{ccc|c} 1 & 2 & 9 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$
(Step 2b)

$\xrightarrow{\quad R_1 = R_1 - 2R_2 \quad}$
$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$
(Step 2c)

The row reduced matrix
$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right)$$
corresponds to the
linear system
$$\begin{cases} x + 5z = 1 \\ y + 2z = -1 \end{cases}$$

The system

$$x + 5z = 1$$
$$y + 2z = -1$$

comes from a matrix in reduced row echelon form. Are we done? Is the system solved?

Yes! Rewrite:

$$x = 1 - 5z$$
$$y = -1 - 2z$$

For any value of z, there is exactly one value of x and y that makes the equations true. But z can be *anything we want*!

So we have found the solution set: it is all values x, y, z where

$$x = 1 - 5z$$
$$y = -1 - 2z \qquad \text{for } z \text{ any real number.}$$
$$(z = z)$$

This is called the **parametric form** for the solution. [interactive picture]

For instance, $(1, -1, 0)$ and $(-4, -3, 1)$ are solutions.

Definition

Consider a *consistent* linear system of equations in the variables $x_1, \ldots, x_n$. Let A be a row echelon form of the matrix for this system.

We say that x_i is a **free variable** if its corresponding column in A is *not* a pivot column.

Important

1. You can choose *any value* for the free variables in a (consistent) linear system.
2. Free variables come from *columns without pivots* in a matrix in row echelon form.

In the previous example, z was free because the reduced row echelon form matrix was

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 2 & -1 \end{array} \right).$$

In this matrix:

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array} \right)$$

the free variables are x_2 and x_4. (What about the last column?)

One More Example

The reduced row echelon form of the matrix for a linear system in x_1, x_2, x_3, x_4 is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array} \right)$$

The free variables are x_2 and x_4: they are the ones whose columns are *not* pivot columns.

This translates into the system of equations

$$\begin{cases} x_1 & + 3x_4 = & 2 \\ & x_3 + 4x_4 = & -1 \end{cases} \implies \boxed{\begin{aligned} x_1 &= & 2 - 3x_4 \\ x_3 &= & -1 - 4x_4 \end{aligned}}$$

What happened to x_2? What is it allowed to be? Anything! The general solution is

$$(x_1, x_2, x_3, x_4) = (2 - 3x_4, x_2, -1 - 4x_4, x_4)$$

for any values of x_2 and x_4. For instance, $(2, 0, -1, 0)$ is a solution $(x_2 = x_4 = 0)$, and $(5, 1, 3, -1)$ is a solution $(x_2 = 1, x_4 = -1)$.

> The boxed equation is called the **parametric form** of the general solution to the system of equations. It is obtained by moving all free variables to the right-hand side of the $=$.

Yet Another Example

The linear system

$$x + y + z = 1 \quad \text{has matrix form} \quad \begin{pmatrix} 1 & 1 & 1 & | & 1 \end{pmatrix}.$$

This is in reduced row echelon form. The free variables are y and z. The parametric form of the general solution is

$$x = 1 - y - z.$$

Rearranging:

$$(x, y, z) = (1 - y - z, \ y, \ z),$$

where y and z are arbitrary real numbers. This was an example in the second lecture!

[interactive]

Poll

Is it possible for a system of linear equations to have exactly two solutions?

Trichotomy

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.

   In this case, the system is *inconsistent*. There are *zero* solutions, i.e. the solution set is *empty*. Picture:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Every column except the last column is a pivot column.

   In this case, the system has a *unique solution*. Picture:

$$\begin{pmatrix} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{pmatrix}$$

3. The last column is not a pivot column, and some other column isn't either.

   In this case, the system has *infinitely many* solutions, corresponding to the infinitely many possible values of the free variable(s). Picture:

$$\begin{pmatrix} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{pmatrix}$$

Summary

- **Row reduction** is an algorithm for solving a system of linear equations represented by an augmented matrix.
- The goal of row reduction is to put a matrix into **(reduced) row echelon form**, which is the "solved" version of the matrix.
- An augmented matrix corresponds to an inconsistent system if and only if there is a pivot in the augmented column.
- Columns without pivots in the RREF of a matrix correspond to **free variables**. You can assign any value you want to the free variables, and you get a unique solution.
- A linear system has zero, one, or infinitely many solutions.
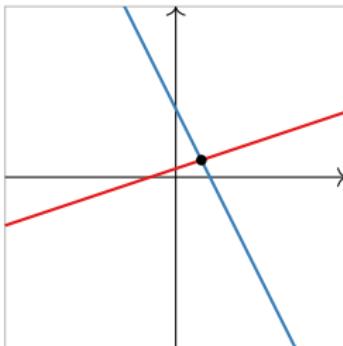
Chapter 2

Systems of Linear Equations: Geometry

Motivation

We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

$$x - 3y = -3$$
$$2x + y = 8$$

This will give us better insight into the properties of systems of equations and their solution sets.

Remember: I expect you to be able to draw pictures!

Section 2.1
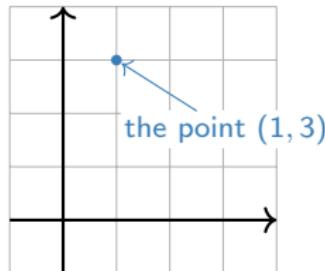
Vectors

Points and Vectors

We have been drawing elements of $\mathbf{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.
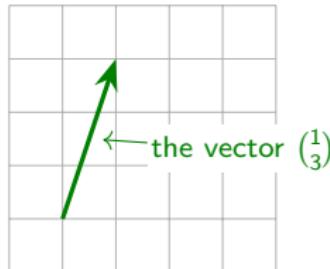
Definition

A **point** is an element of $\mathbf{R}^n$, drawn as a point (a dot).

the point $(1, 3)$

A **vector** is an element of $\mathbf{R}^n$, drawn as an arrow. When we think of an element of $\mathbf{R}^n$ as a vector, we'll usually write it vectically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

[interactive]

the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The difference is purely psychological: *points and vectors are just lists of numbers*.

Points and Vectors

So why make the distinction?

A vector need not start at the origin: *it can be located anywhere*! In other words, an arrow is determined by its length and its direction, not by its location.

These arrows all represent the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

However, unless otherwise specified, we'll assume a vector starts at the origin.

Vector Algebra

Definition

- We can add two vectors together:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a + x \\ b + y \\ c + z \end{pmatrix}.$$

- We can multiply, or **scale**, a vector by a real number c:
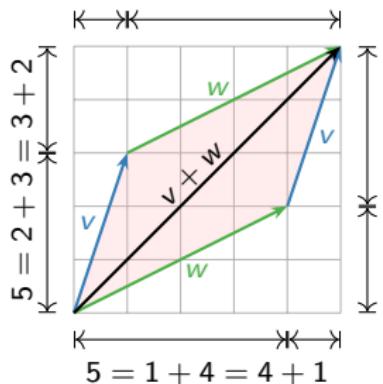
$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \cdot x \\ c \cdot y \\ c \cdot z \end{pmatrix}.$$

We call c a **scalar** to distinguish it from a vector. If v is a vector and c is a scalar, cv is called a **scalar multiple** of v.

(And likewise for vectors of length n.) For instance,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \quad \text{and} \quad -2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -6 \end{pmatrix}.$$

Vector Addition and Subtraction: Geometry

The parallelogram law for vector addition

Geometrically, the sum of two vectors v, w is obtained as follows: place the tail of w at the head of v. Then $v + w$ is the vector whose tail is the tail of v and whose head is the head of w. Doing this both ways creates a parallelogram. For example,

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Why? The width of $v + w$ is the sum of the widths, and likewise with the heights.   [interactive]
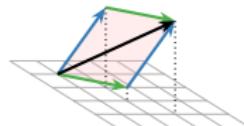
Vector subtraction

Geometrically, the difference of two vectors v, w is obtained as follows: place the tail of v and w at the same point. Then $v - w$ is the vector from the head of w to the head of v. For example,

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

Why? If you add $v - w$ to w, you get v. [interactive]

This works in higher dimensions too!
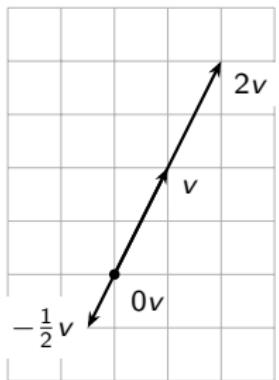
# Scalar Multiplication: Geometry

Scalar multiples of a vector
These have the same *direction* but a different *length*.

Some multiples of v.

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$2v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$-\frac{1}{2}v = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$

$$0v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

All multiples of v.

[interactive]

So the scalar multiples of v form a *line*.

Linear Combinations

We can add and scalar multiply in the same equation:
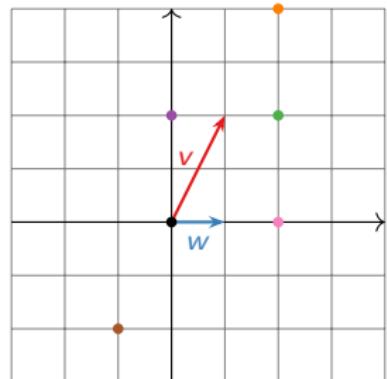
$$w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

where $c_1, c_2, \ldots, c_p$ are scalars, $v_1, v_2, \ldots, v_p$ are vectors in $\mathbf{R}^n$, and w is a vector in $\mathbf{R}^n$.

Definition
We call w a **linear combination** of the vectors $v_1, v_2, \ldots, v_p$. The scalars $c_1, c_2, \ldots, c_p$ are called the **weights** or **coefficients**.

Example

Let $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

What are some linear combinations of v and w?

- ▶ $v + w$
- ▶ $v - w$
- ▶ $2v + 0w$
- ▶ $2w$
- ▶ $-v$

[interactive: 2 vectors]   [interactive: 3 vectors]
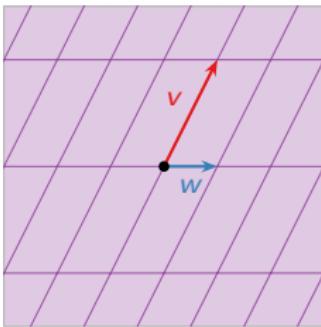
Poll

Poll
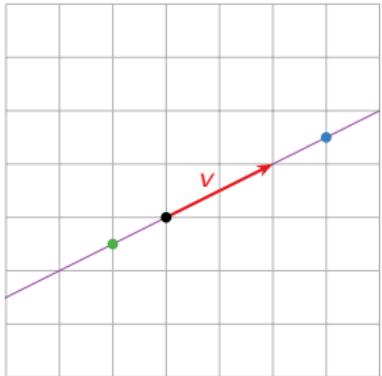
Is there any vector in $\mathbf{R}^2$ that is *not* a linear combination of v and w?

No: in fact, *every* vector in $\mathbf{R}^2$ is a combination of v and w.

(The purple lines are to help measure *how much* of v and w you need to get to a given point.)

More Examples
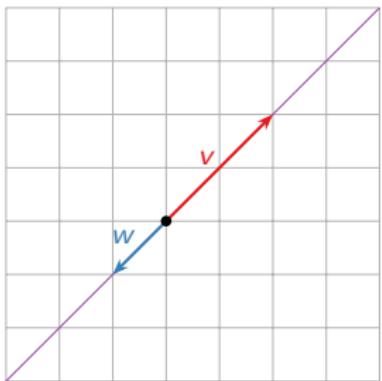
What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

- ▶ $\frac{3}{2}v$
- ▶ $-\frac{1}{2}v$
- ▶ ...

What are *all* linear combinations of v?

All vectors cv for c a real number. I.e., all *scalar multiples* of v. These form a *line*.

Question

What are all linear combinations of

$$v = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}?$$

Answer: The line which contains both vectors.

What's different about this example and the one on the poll? [interactive]

Section 2.2

Vector Equations and Spans

Solve the following system of linear equations:

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3. \end{aligned}$$

We can write all three equations at once as vectors:

$$\begin{pmatrix} x - y \\ 2x - 2y \\ 6x - y \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

We can write this as a linear combination:

$$x \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + y \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}.$$

So we are asking:

Question: Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

$$\begin{aligned} x - y &= 8 \\ 2x - 2y &= 16 \\ 6x - y &= 3 \end{aligned}$$

$\xrightarrow{\text{matrix form}}$

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

$\xrightarrow{\text{row reduce}}$

$$\left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right)$$

$\xrightarrow{\text{solution}}$

$$\begin{aligned} x &= -1 \\ y &= -9 \end{aligned}$$

Conclusion:

$$-\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

[interactive] $\longleftarrow$ (this is the picture of a *consistent* linear system)

What is the relationship between the vectors in the linear combination and the matrix form of the linear equation? They have the same columns!

Shortcut: You can go directly between augmented matrices and vector equations.

Vector Equations and Linear Equations

Summary

The **vector equation**

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b,$$

where $v_1, v_2, \ldots, v_p, b$ are vectors in $\mathbf{R}^n$ and $x_1, x_2, \ldots, x_p$ are scalars, has the same solution set as the linear system with augmented matrix

$$\begin{pmatrix} | & | & & | & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | & | \end{pmatrix},$$

where the v_i's and b are the columns of the matrix.

So we now have (at least) *two* equivalent ways of thinking about linear systems of equations:

1. Augmented matrices.
2. Linear combinations of vectors (vector equations).

The last one is more geometric in nature.

Span

It is important to know what are *all* linear combinations of a set of vectors $v_1, v_2, \ldots, v_p$ in $\mathbf{R}^n$: it's exactly the collection of all b in $\mathbf{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \ldots, x_p$)

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution (i.e., is consistent).

Definition

"the set of"   "such that"

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$. The **span** of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\text{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

$$\text{Span}\{v_1, v_2, \ldots, v_p\} = \left\{\, x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbf{R} \,\right\}.$$

Synonyms: $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset **spanned by** or **generated by** $v_1, v_2, \ldots, v_p$.

> This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!

Now we have several equivalent ways of making the same statement:

1. A vector b is in the span of $v_1, v_2, \ldots, v_p$.

2. The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution.

3. The linear system with augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_p & b \\ | & | & & | & | \end{array} \right)$$
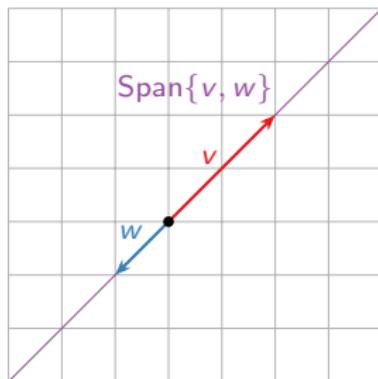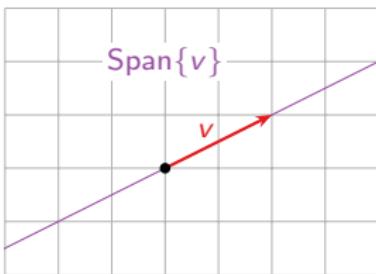
is consistent.

[interactive example] $\longleftarrow$ (this is the picture of an *inconsistent* linear system)

Note: **equivalent** means that, for any given list of vectors $v_1, v_2, \ldots, v_p, b$, *either* all three statements are true, *or* all three statements are false.

Pictures of Span

Drawing a picture of $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.
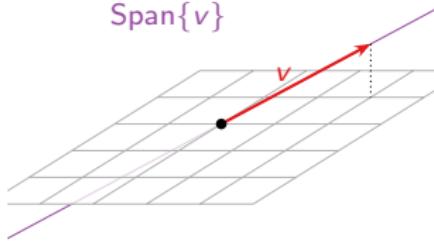
[interactive: span of two vectors in $\mathbf{R}^2$]
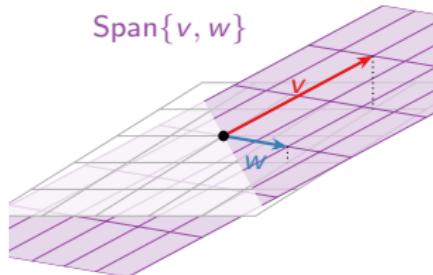
Pictures of Span

In $\mathbf{R}^3$
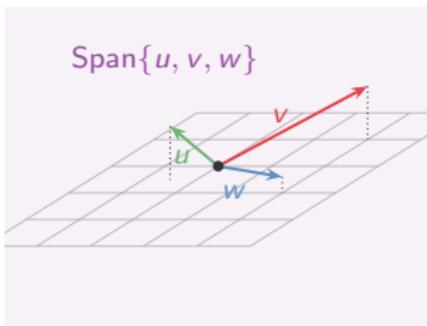
Span$\{v\}$
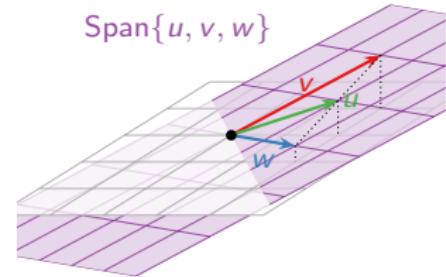
Span$\{v, w\}$

Span$\{u, v, w\}$

Span$\{u, v, w\}$

[interactive: span of two vectors in $\mathbf{R}^3$]   [interactive: span of three vectors in $\mathbf{R}^3$]

Poll

Poll

How many vectors are in Span $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$?

A. Zero

B. One

C. Infinity

In general, it appears that $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the smallest "linear space" (line, plane, etc.) containing the origin and all of the vectors $v_1, v_2, \ldots, v_p$.

We will make this precise later.

Summary

The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbf{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.

Section 2.3

Matrix Equations

Matrix $\times$ Vector

the first number is
the number of rows

the second number is
the number of columns

Let A be an $m \times n$ matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{with columns } v_1, v_2, \ldots, v_n$$

Definition

The **product** of A with a vector x in $\mathbf{R}^n$ is the linear combination

this means the equality
is a *definition*

$$Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{def}}{=} x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

these must be equal

The output is a vector in $\mathbf{R}^m$.

Note that the number of columns of A has to equal the number of rows of x.

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Question

Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. How can you write the vector equation

$$2v_1 + 3v_2 - 4v_3 = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

in terms of matrix multiplication?

Answer: Let A be the matrix with colums v_1, v_2, v_3, and let x be the vector with entries $2, 3, -4$. Then

$$Ax = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 2v_1 + 3v_2 - 4v_3,$$

so the vector equation is equivalent to the matrix equation

$$Ax = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}.$$

Let $v_1, v_2, \ldots, v_n$, and b be vectors in $\mathbf{R}^m$. Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b.$$

It is equivalent to the **matrix equation**

$$Ax = b$$

where

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Conversely, if A is any $m \times n$ matrix, then

$$Ax = b \qquad \text{is equivalent to the vector equation} \qquad x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = b$$

where $v_1, \ldots, v_n$ are the columns of A, and $x_1, \ldots, x_n$ are the entries of x.

Linear Systems, Vector Equations, Matrix Equations, ...

We now have *four* equivalent ways of writing (and thinking about) linear systems:

1. As a system of equations:

$$2x_1 + 3x_2 = 7$$
$$x_1 - x_2 = 5$$

2. As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array} \right)$$

3. As a vector equation ($x_1 v_1 + \cdots + x_n v_n = b$):

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation ($Ax = b$):

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

We will move back and forth freely between these over and over again, for the rest of the semester. Get comfortable with them now!

In particular, *all four have the same solution set.*

Definition

A **row vector** is a matrix with one row. The product of a row vector of length n and a (column) vector of length n is

$$
\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}
\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
\overset{\text{def}}{=} a_1 x_1 + \cdots + a_n x_n.
$$

This is a scalar.

If A is an $m \times n$ matrix with rows $r_1, r_2, \ldots, r_m$, and x is a vector in $\mathbf{R}^n$, then

$$
Ax = \begin{pmatrix} - r_1 - \\ - r_2 - \\ \vdots \\ - r_m - \end{pmatrix} x = \begin{pmatrix} r_1 x \\ r_2 x \\ \vdots \\ r_m x \end{pmatrix}
$$

This is a vector in $\mathbf{R}^m$ (again).

Matrix × Vector
Both ways

Example

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} (4\,5\,6)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ (7\,8\,9)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \\ 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

Note this is the same as before:

$$\begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 4 \\ 7 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 8 \end{pmatrix} + 3 \begin{pmatrix} 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \end{pmatrix} = \begin{pmatrix} 32 \\ 50 \end{pmatrix}.$$

> Now you have *two* ways of computing Ax.
>
> In the second, you calculate Ax one entry at a time.
>
> The second way is usually the most convenient, but we'll use both.

In engineering, the first way corresponds to "superposition of states", and the second is "taking a measurement".

Spans and Solutions to Equations

Let A be a matrix with columns $v_1, v_2, \ldots, v_n$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

Very Important Fact That Will Appear on Every Midterm and the Final

$Ax = b$ has a solution

$$\Longleftrightarrow \text{ there exist } x_1, \ldots, x_n \text{ such that } A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b$$

"if and only if"

$\Longleftrightarrow$ there exist $x_1, \ldots, x_n$ such that $x_1 v_1 + \cdots + x_n v_n = b$

$\Longleftrightarrow$ b is a linear combination of $v_1, \ldots, v_n$

$\Longleftrightarrow$ b is in the span of the columns of A.

The last condition is geometric.

Spans and Solutions to Equations
Example

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?
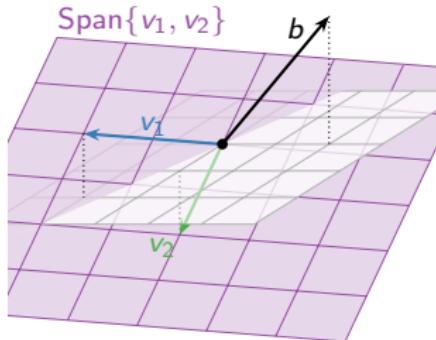
[interactive]

Columns of A:

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Target vector:

$$b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A? It sure doesn't look like it.

Conclusion: $Ax = b$ is *inconsistent*.

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's check by solving the matrix equation using row reduction.

The first step is to put the system into an augmented matrix.

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

The last equation is $0 = 1$, so the system is *inconsistent*.

In other words, the matrix equation

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

has no solution, as the picture shows.

Spans and Solutions to Equations
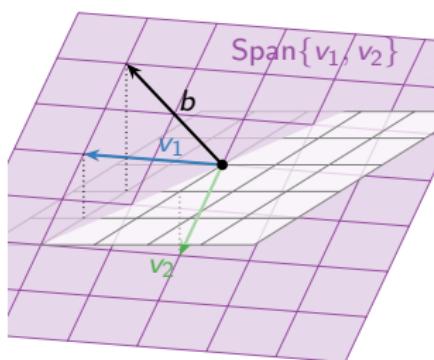
Example

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

[interactive]

Columns of A:

$$v_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Target vector:

$$b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

Is b contained in the span of the columns of A? It looks like it: in fact,

$$b = 1v_1 + (-1)v_2 \implies x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Question

Let $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}$. Does the equation $Ax = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ have a solution?

Answer: Let's do this systematically using row reduction.

$$\left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

This gives us

$$x = 1 \qquad y = -1.$$

This is consistent with the picture on the previous slide:

$$1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \text{or} \qquad A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Poll

Which of the following true statements can be checked by eye-balling them, *without* row reduction?

A. $\begin{pmatrix} 3 & 0 & 0 & | & 0 \\ 3 & 10 & -1 & | & 1 \\ 4 & 20 & -2 & | & 2 \end{pmatrix}$ is consistent.

B. $\begin{pmatrix} 3 & 0 & 0 & | & 0 \\ 3 & 5 & 6 & | & 1 \\ 4 & 7 & 8 & | & 2 \end{pmatrix}$ is consistent.

C. $\begin{pmatrix} 3 & 0 & 0 & | & 0 \\ 3 & 1 & 0 & | & 1 \\ 4 & 0 & \sqrt{2} & | & 2 \end{pmatrix}$ is consistent.

D. $\begin{pmatrix} 5 & 6 & 3 & | & 0 \\ 7 & 8 & 3 & | & 1 \\ 0 & 0 & 4 & | & 2 \end{pmatrix}$ is consistent.

When Solutions Always Exist

Here are criteria for a linear system to *always* have a solution.

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution *for all b in $\mathbf{R}^m$*.

2. The span of the columns of A is all of $\mathbf{R}^m$.

3. A has a pivot in each row.

recall that this means that for given A, either they're all true, or they're all false

Why is (1) the same as (2)? This was the $\boxed{\text{Very Important}}$ box from before.

Why is (1) the same as (3)? If A has a pivot in each row then its reduced row echelon form looks like this:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix}$$

and $(\,A \mid b\,)$ reduces to this:

$$\left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & \star \\ 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & \star \end{array}\right).$$

There's no b that makes it inconsistent, so there's always a solution. If A doesn't have a pivot in each row, then its reduced form looks like this:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and this can be made inconsistent:

$$\left(\begin{array}{ccccc|c} 1 & 0 & \star & 0 & \star & 0 \\ 0 & 1 & \star & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{array}\right).$$

Theorem

Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:

1. $Ax = b$ has a solution *for all b* in $\mathbf{R}^m$.
2. The span of the columns of A is all of $\mathbf{R}^m$.
3. A has a pivot in each row.

In the following demos, the violet region is the span of the columns of A. This is the same as the set of all b such that $Ax = b$ has a solution.

[example where the criteria are satisfied]

[example where the criteria are not satisfied]

Let c be a scalar, u, v be vectors, and A a matrix.

- $A(u + v) = Au + Av$
- $A(cv) = cAv$

For instance, $A(3u - 7v) = 3Au - 7Av$.

Consequence: If u and v are solutions to $Ax = 0$, then so is every vector in Span$\{u, v\}$. Why?

$$\begin{cases} Au = 0 \\ Av = 0 \end{cases} \implies A(xu + yv) = xAu + yAv = x0 + y0 = 0.$$

(Here 0 means the zero vector.)

Important

The set of solutions to $Ax = 0$ is a span.

Summary
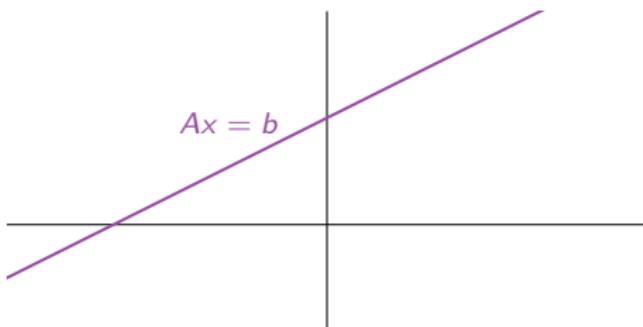
- We have four equivalent ways of writing a system of linear equations:
  1. As a system of equations.
  2. As an augmented matrix.
  3. As a vector equation.
  4. As a matrix equation $Ax = b$.
- $Ax = b$ is consistent if and only if b is in the span of the columns of A. The latter condition is geometric: you can draw pictures of it.
- $Ax = b$ is consistent for all b in $\mathbf{R}^m$ if and only if the columns of A span $\mathbf{R}^m$.

Section 2.4

Solution Sets

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations $Ax = b$, using spans.

Recall: the **solution set** is the collection of all vectors x such that $Ax = b$ is true.

Last time we discussed the set of vectors b for which $Ax = b$ has a solution.

We also described this set using spans, but it was a *different problem*.

Everything is easier when $b = 0$, so we start with this case.

Definition
A system of linear equations of the form $Ax = 0$ is called **homogeneous.**

These are linear equations where everything to the right of the $=$ is zero. The opposite is:

Definition
A system of linear equations of the form $Ax = b$ with $b \neq 0$ is called **inhomogeneous.**

A homogeneous system always has the solution $x = 0$. This is called the **trivial solution.** The nonzero solutions are called **nontrivial.**

> **Observation**
>
> $Ax = 0$ has a nontrivial solution
> $$\iff \text{ there is a free variable}$$
> $$\iff A \text{ has a column with no pivot.}$$

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

We know how to do this: first form an augmented matrix and row reduce.

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right).$$

The only solution is the trivial solution $x = 0$.

Observation

Since the last column (everything to the right of the $=$) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} \quad x_1 - 3x_2 = 0$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = 3x_2 \\ x_2 = x_2 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

This last equation is called the **parametric vector form** of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.
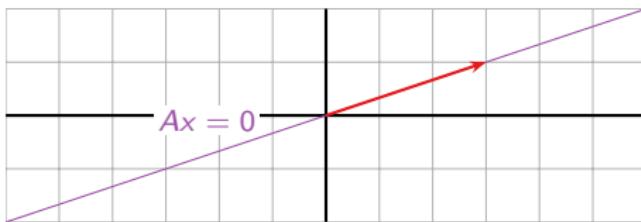
### Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for any x_2 in $\mathbf{R}$. The solution set is $\text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$.

$Ax = 0$

[interactive]

Note: *one* free variable means the solution set is a *line* in $\mathbf{R}^2$ ($2 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} x_1 - x_2 + 2x_3 = 0$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = x_2 - 2x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$
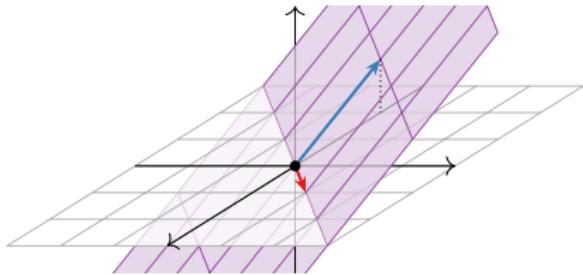
Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

Answer: Span $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

[interactive]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^3$ ($3 = \#$ variables $= \#$ columns).

Question

What is the solution set of $Ax = 0$, where $A =$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 & - 8x_3 - 7x_4 = 0 \\ & x_2 + 4x_3 + 3x_4 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = & 8x_3 + 7x_4 \\ x_2 = & -4x_3 - 3x_4 \\ x_3 = & x_3 \\ x_4 = & x_4 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

Question

What is the solution set of $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer: $\text{Span}\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$

[not pictured here]

Note: *two* free variables means the solution set is a *plane* in $\mathbf{R}^4$ ($4 = \#$ variables $= \#$ columns).

Let A be an $m \times n$ matrix. Suppose that the free variables in the homogeneous equation $Ax = 0$ are, for example, x_3, x_6, and x_8.

1. Find the reduced row echelon form of A.
2. Write the parametric form of the solution set, including the redundant equations $x_3 = x_3$, $x_6 = x_6$, and $x_8 = x_8$. Put equations for all of the x_i in order.
3. Make a single vector equation from these equations by putting x_3, x_6, and x_8 as coefficients of vectors v_3, v_6, and v_8, respectively.

The solutions to $Ax = 0$ will then be expressed in the form

$$x = x_3 v_3 + x_6 v_6 + x_8 v_8$$

for some vectors v_3, v_6, v_8 in $\mathbf{R}^n$, and any scalars x_3, x_6, x_8.

In this case, the solution set to $Ax = 0$ is

$$\mathrm{Span}\{v_3,\ v_6,\ v_8\}.$$

The equation above is called the **parametric vector form** of the solution.

We emphasize the fact that **the set of solutions to $Ax = 0$ is a span**.

Poll

The trivial solution is always a solution to a homogeneous system, so answer A is impossible.

This matrix has only one solution to $Ax = 0$:   [interactive]

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This matrix has infinitely many solutions to $Ax = 0$:   [interactive]

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -3 & -3 \\ 2 & -6 & -6 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equation}} x_1 - 3x_2 = -3$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 3x_2 - 3 \\ x_2 = x_2 + 0 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

The only difference from the homogeneous case is the constant vector $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

Note that p is itself a solution: take $x_2 = 0$.
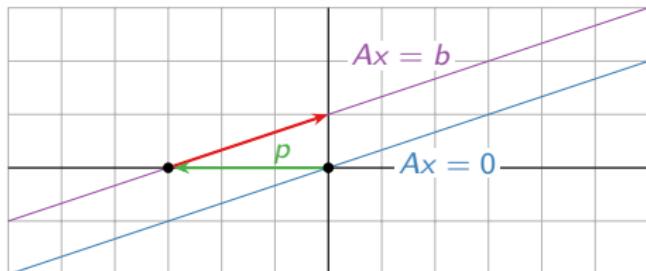
Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

Answer: $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ for any x_2 in $\mathbf{R}$.

This is a *translate* of $\text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$: it is the parallel line through $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

It can be written

$$\text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

[interactive]

Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -2 & 2 & -4 & -2 \end{array} \right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\xrightarrow{\text{equation}} \quad x_1 - x_2 + 2x_3 = 1$$

$$\xrightarrow{\text{parametric form}} \quad \begin{cases} x_1 = x_2 - 2x_3 + 1 \\ x_2 = x_2 \\ x_3 = x_3 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$
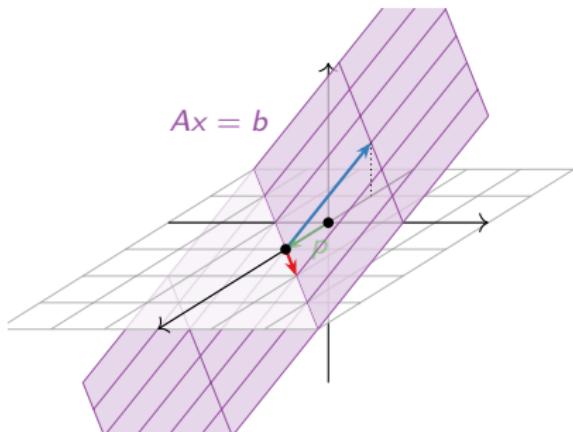
Question

What is the solution set of $Ax = b$, where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

Answer: $\text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$

$Ax = b$

The solution set is a *translate* of

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}:$$

it is the parallel plane through

$$p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

[interactive]
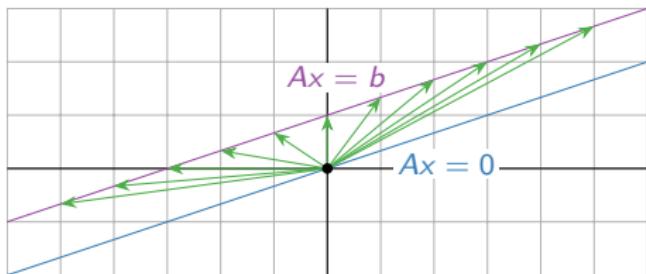
> **Key Observation**
>
> The set of solutions to $Ax = b$, if it is nonempty, is obtained by taking one **specific** or **particular solution** p to $Ax = b$, and adding all solutions to $Ax = 0$.

Why? If $Ap = b$ and $Ax = 0$, then

$$A(p + x) = Ap + Ax = b + 0 = b,$$

so $p + x$ is also a solution to $Ax = b$.

We know the solution set of $Ax = 0$ is a span. So the solution set of $Ax = b$ is a *translate* of a span: it is *parallel* to a span. (Or it is empty.)

This works for *any* specific solution p: it doesn't have to be the one produced by finding the parametric vector form and setting the free variables all to zero, as we did before.

[interactive 1]   [interactive 2]

Solution Sets and Spans of Columns

Very Important

Let A be an $m \times n$ matrix. There are now two *completely different* things you know how to describe using spans:

- The **solution set:** for fixed b, this is all x such that $Ax = b$.
  - This is a span if $b = 0$, or a translate of a span in general (if it's consistent).
  - Lives in $\mathbf{R}^n$.
  - Computed by finding the parametric vector form.

- The **span of the columns:** this is all b such that $Ax = b$ is consistent.
  - This is the span of the columns of A.
  - Lives in $\mathbf{R}^m$.

Don't confuse these two geometric objects!

Much of the first midterm tests whether you understand both.

[interactive]

Summary
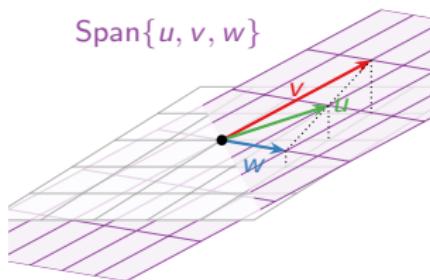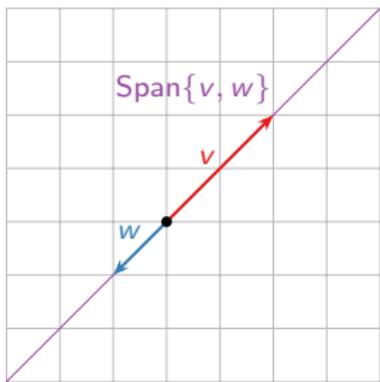
- The solution set to a **homogeneous** system $Ax = 0$ is a span. It always contains the **trivial solution** $x = 0$.
- The solution set to a **nonhomogeneous** system $Ax = b$ is either empty, or it is a translate of a span: namely, it is a translate of the solution set of $Ax = 0$.
- The solution set to $Ax = b$ can be expressed as a translate of a span by computing the **parametric vector form** of the solution.
- The solution set to $Ax = b$ and the span of the columns of A (from the previous lecture) are two completely different things, and you have to understand them separately.

Section 2.5

Linear Independence

Motivation

Sometimes the span of a set of vectors is "smaller" than you expect from the number of vectors.

This means that (at least) one of the vectors is *redundant*: you're using "too many" vectors to describe the span.

Notice in each case that one vector in the set is already in the span of the others—so it doesn't make the span bigger.

Today we will formalize this idea in the concept of *linear (in)dependence*.

Linear Independence

Definition
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

In other words, $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \ldots, x_p$, not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0.$$

This is called a **linear dependence relation** or an **equation of linear dependence**.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

> Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial solution? How do we solve this kind of vector equation?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So $x = -2z$ and $y = -z$. So the vectors are linearly *dependent*, and an equation of linear dependence is (taking $z = 1$)

$$-2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

[interactive]

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial solution?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The trivial solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the unique solution. So the vectors are

linearly *in*dependent.

[interactive]

Linear Independence and Matrix Columns

In general, $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution, if and only if the matrix equation

$$Ax = 0$$

has only the trivial solution, where A is the matrix with columns $v_1, v_2, \ldots, v_p$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix A has a pivot in each column.

Important

- The vectors $v_1, v_2, \ldots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \ldots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \ldots, v_p$ of A are linearly independent, or will produce a linear dependence relation.

Linear Independence

Suppose that one of the vectors $\{v_1, v_2, \ldots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Then the vectors are linearly *de*pendent:

$$2v_1 - \frac{1}{2}v_2 - v_3 + 6v_4 = 0.$$

Conversely, if the vectors are linearly dependent

$$2v_1 - \frac{1}{2}v_2 + 6v_4 = 0.$$

then one vector is a linear combination of (in the span of) the other ones:

$$v_2 = 4v_1 + 12v_4.$$

Theorem
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Equivalently:

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if you can remove one of the vectors without shrinking the span.

Indeed, if $v_2 = 4v_1 + 12v_3$, then a linear combination of v_1, v_2, v_3 is

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = x_1 v_1 + x_2(4v_1 + 12v_3) + x_3 v_3$$
$$= (x_1 + 4x_2)v_1 + (12x_2 + x_3)v_3,$$

which is already in $\text{Span}\{v_1, v_3\}$.

Conclusion: v_2 was redundant.

Theorem
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Better Theorem
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

Equivalently, $\{v_1, v_2, \ldots, v_p\}$ is linearly *in*dependent if for every j, the vector v_j is not in $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

This means $\text{Span}\{v_1, v_2, \ldots, v_j\}$ is *bigger* than $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

> **Translation**
>
> A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

Better Theorem
A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

Why? Take the largest j such that v_j is in the span of the others. Then v_j is in the span of $v_1, v_2, \ldots, v_{j-1}$. Why? If not ($j = 3$):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$
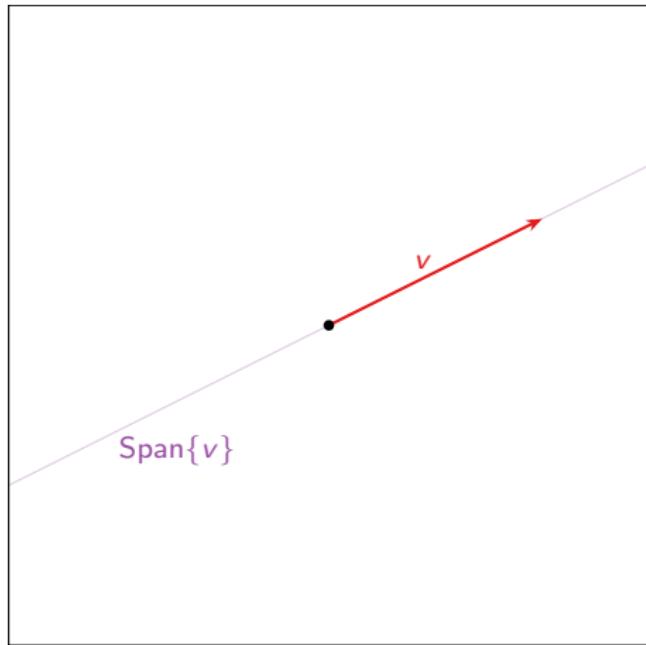
Rearrange:

$$v_4 = -\frac{1}{6}\left(2v_1 - \frac{1}{2}v_2 - v_3\right)$$

so v_4 works as well, but v_3 was supposed to be the last one that was in the span of the others.
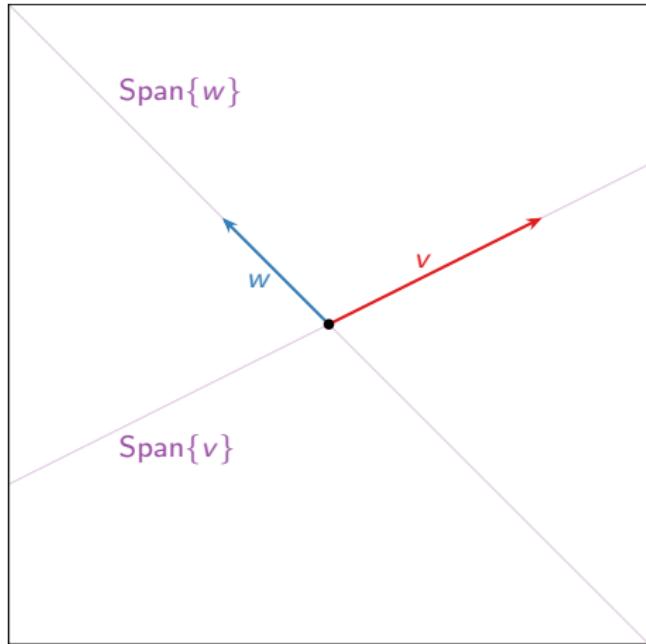
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Span$\{v\}$

v

[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

Linear Independence
Pictures in $\mathbf{R}^2$

[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]
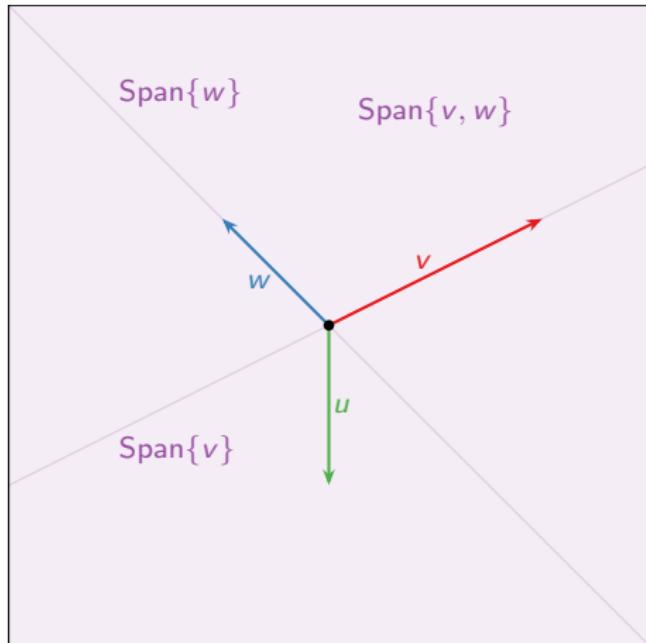
One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent
- Neither is in the span of the other.
- Span got bigger.

[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent
  ▶ Neither is in the span of the other.
 ▶ Span got bigger.
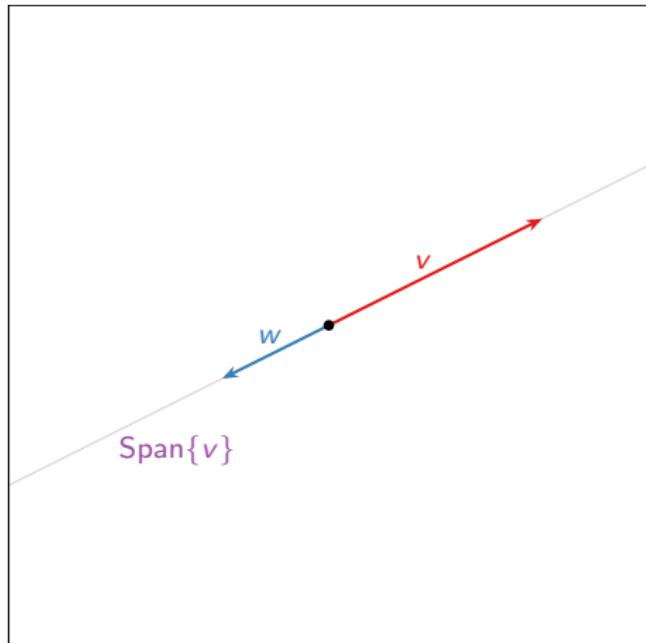
Three vectors $\{v, w, u\}$:
Linearly dependent:
  ▶ u is in Span$\{v, w\}$.
  ▶ Span didn't get bigger after adding u.
  ▶ Can remove u without shrinking the span.
Also v is in Span$\{u, w\}$ and w is in Span$\{u, v\}$.

Linear Independence

Two collinear vectors $\{v, w\}$:
Linearly dependent:

- w is in Span$\{v\}$.
- Can remove w without shrinking the span.
- Span didn't get bigger when we added w.
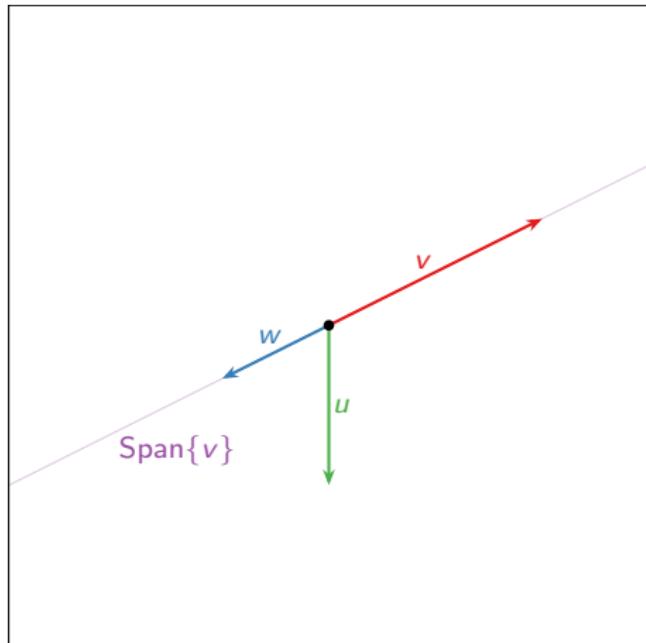
Observe: *Two* vectors are linearly *de*pendent if and only if they are *collinear*.

[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

Linear Independence
Pictures in $\mathbf{R}^2$

[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

Three vectors $\{v, w, u\}$:
Linearly dependent:

- w is in $\mathrm{Span}\{u, v\}$.
- Can remove w without shrinking the span.
- Span didn't get bigger when we added w.
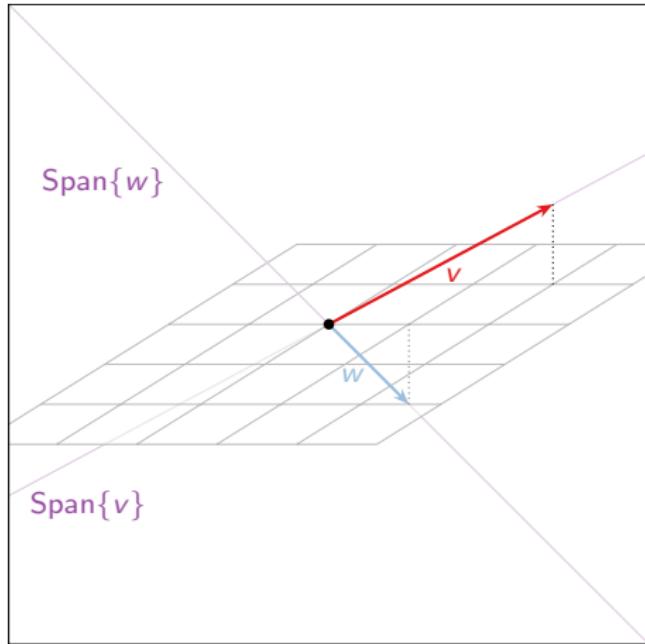
Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!

Linear Independence

Span$\{w\}$

v

w

Span$\{v\}$

[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Two vectors $\{v, w\}$:
Linearly independent:
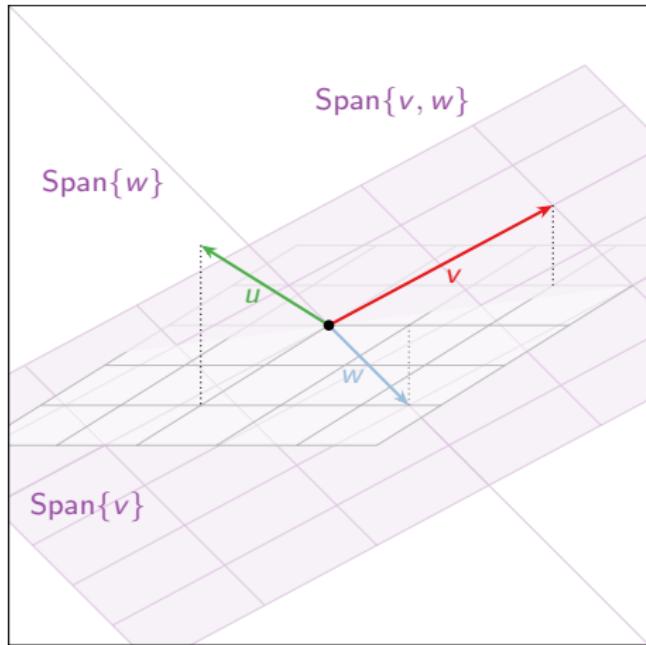
- Neither is in the span of the other.
- Span got bigger when we added w.

Linear Independence
Pictures in **R**^{3}

Span{v, w}

Span{w}

u

v

w

Span{v}

Three vectors $\{v, w, u\}$:
Linearly independent: span got bigger when we added u.

[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Linear Independence
Pictures in $\mathbf{R}^3$

[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Three vectors $\{v, w, x\}$:
Linearly dependent:

- x is in Span$\{v, w\}$.
- Can remove x without shrinking the span.
- Span didn't get bigger when we added x.

Poll

Poll

Are there four vectors u, v, w, x in $\mathbf{R}^3$ which are linearly dependent, but such that u is *not* a linear combination of v, w, x? If so, draw a picture; if not, give an argument.

Yes: actually the pictures on the previous slides provide such an example.

Linear dependence of $\{v_1, \ldots, v_p\}$ means *some* v_i is a linear combination of the others, not *any*.

Linear Dependence and Free Variables

Theorem
Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$, and consider the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

Then you can delete the columns of A *without* pivots (the columns corresponding to free variables), without changing $\text{Span}\{v_1, v_2, \ldots, v_p\}$. The pivot columns are linearly independent, so you can't delete any more columns.

This means that each time you add a pivot column, then the span increases.

Upshot

Let d be the number of pivot columns in the matrix A above.
- ▶ If $d = 1$ then $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a line.
- ▶ If $d = 2$ then $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a plane.
- ▶ If $d = 3$ then $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a 3-space.
- ▶ Etc.

Why? If the matrix is in RREF:

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the column without a pivot is in the span of the pivot columns:

$$\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the pivot columns are linearly independent:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \implies x_1 = x_2 = x_4 = 0.$$

Linear Dependence and Free Variables

Justification

Why? If the matrix is not in RREF, then row reduce:

$$A = \begin{pmatrix} 1 & 7 & 23 & 3 \\ 2 & 4 & 16 & 0 \\ -1 & -2 & -8 & 4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The following vector equations have the same solution set:

$$x_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 23 \\ 16 \\ -8 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 0$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

We conclude that $\begin{pmatrix} 23 \\ 16 \\ -8 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

and that $x_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 0$ has only the trivial solution.

Fact 1: Say $v_1, v_2, \ldots, v_n$ are in $\mathbf{R}^m$. If $n > m$ then $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in $\mathbf{R}^3$, for instance.

> A wide matrix can't have linearly independent columns.

Fact 2: If one of $v_1, v_2, \ldots, v_n$ is zero, then $\{v_1, v_2, \ldots, v_n\}$ is linearly dependent. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

> A set containing the zero vector is linearly dependent.

Summary

- A set of vectors is **linearly independent** if removing one of the vectors shrinks the span; otherwise it's **linearly dependent**.
- There are several other criteria for linear (in)dependence which lead to pretty pictures.
- The columns of a matrix are linearly independent if and only if the RREF of the matrix has a pivot in every *column*.
- The pivot columns of a matrix A are linearly independent, and you can delete the non-pivot columns (the "free" columns) without changing the span of the columns.
- Wide matrices cannot have linearly independent columns.

> **Warning**
>
> These are not the official definitions!

Section 2.6
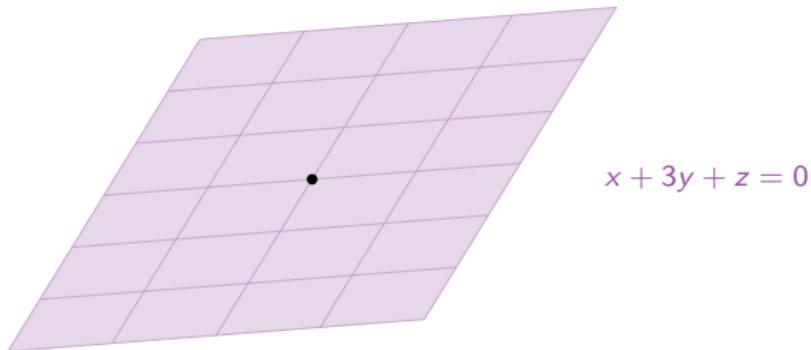
Subspaces

Today we will discuss **subspaces** of $\mathbf{R}^n$.

A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbf{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.

$x + 3y + z = 0$

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V.   "not empty"
2. If u and v are in V, then $u + v$ is also in V.   "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V.   "closed under $\times$ scalars"

> **Fast-forward**
>
> Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven't computed what those vectors are yet.

Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V.   "not empty"
2. If u and v are in V, then $u + v$ is also in V.   "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V.   "closed under $\times$ scalars"

What does this mean?

- If v is in V, then all scalar multiples of v are in V by (3). In other words, the line through any nonzero vector in V is also in V.

- If u, v are in V, then cu and dv are in V for any scalars c, d by (3). So $cu + dv$ is in V by (2). So Span$\{u, v\}$ is contained in V.

- Likewise, if $v_1, v_2, \ldots, v_n$ are all in V, then Span$\{v_1, v_2, \ldots, v_n\}$ is contained in V: a subspace contains the span of any set of vectors in it.
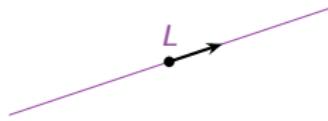
If you pick enough vectors in V, eventually their span will fill up V, so:

> A subspace is a span of some set of vectors in it.

Examples
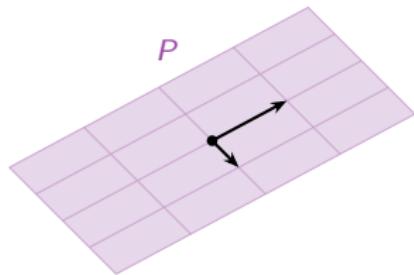
### Example

A line L through the origin is a subspace: L contains zero and is easily seen to be closed under addition and scalar multiplication.

Example

A plane P through the origin is a subspace: P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.

Example

All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example

The subset $\{0\}$: this subspace contains only one vector.

Note these are all pictures of spans! (Line, plane, space, etc.)
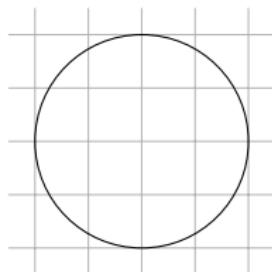
A **subset** of $\mathbf{R}^n$ is any collection of vectors in $\mathbf{R}^n$ whatsoever. For example, the unit circle

$$C = \{(x, y) \text{ in } \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

is a subset of $\mathbf{R}^2$, but it is not a subspace.

All of the following non-examples on the next slide are still subsets.

A **subspace** is a special kind of subset, that satisfies the three defining properties.
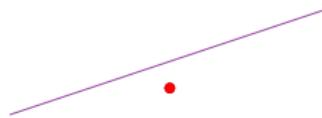
Subset: *yes*
Subspace: *no*
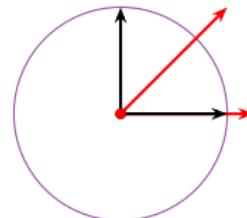
Non-Example

A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.
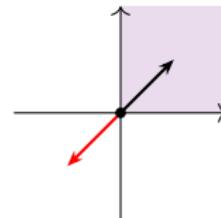
Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."
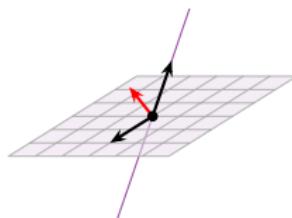
Non-Example

The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.

Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.

Subspaces are Spans, and Spans are Subspaces

Theorem
Any $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a subspace.

> **!!!**
> Every subspace is a span, and every span is a subspace.

Definition
If $V = \text{Span}\{v_1, v_2, \ldots, v_p\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_p$. We call $\{v_1, v_2, \ldots, v_p\}$ a **spanning set** for V.

Check:

1. $0 = 0v_1 + 0v_2 + \cdots + 0v_p$ is in the span.
2. If, say, $u = 3v_1 + 4v_2$ and $v = -v_1 - 2v_2$, then

$$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

   is also in the span.
3. Similarly, if u is in the span, then so is cu for any scalar c.

Poll

Which of the following are subspaces?
  A. The empty set $\{\}$.
  B. The solution set to a homogeneous system of linear equations.
  C. The solution set to an inhomogeneous system of linear equations.
  D. The set of all vectors in $\mathbf{R}^n$ with rational (fraction) coordinates.
For the ones which are not subspaces, which property(ies) do they not satisfy?

A. This is not a subspace: it does not contain the zero vector.
B. This is a subspace: the solution set is a span, produced by finding the parametric vector form of the solution.
C. This is not a subspace: it does not contain 0.
D. This is not a subspace: it is not closed under multiplication by scalars (e.g. by π).
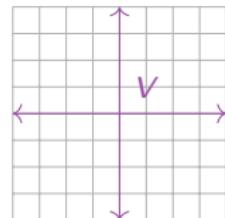
Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \mid ab = 0 \right\}$. Let's check if V is a subspace or not.

1. Does V contain the zero vector? $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies ab = 0$ ✔

3. Is V closed under scalar multiplication?
   ▶ Let $\begin{pmatrix} a \\ b \end{pmatrix}$ be (an unknown vector) in V.
   ▶ *This means:* a and b are numbers such that $ab = 0$.
   ▶ Let c be a scalar. Is $c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$ in V?
   ▶ *This means:* $(ca)(cb) = 0$.
   ▶ Well, $(ca)(cb) = c^2(ab) = c^2(0) = 0$ ✔

2. Is V closed under addition?
   ▶ Let $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} a' \\ b' \end{pmatrix}$ be (unknown vectors) in V.
   ▶ *This means:* $ab = 0$ and $a'b' = 0$.
   ▶ Is $\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a+a' \\ b+b' \end{pmatrix}$ in V?
   ▶ *This means:* $(a + a')(b + b') = 0$.
   ▶ This is not true for all such a, a', b, b': for instance, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are in V, but their sum $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not in V, because $1 \cdot 1 \neq 0$. ✘

We conclude that V is *not* a subspace. A picture is above. (It doesn't look like a span.)

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written Col A.

- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:
$$\text{Nul } A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$
This is a subspace of $\mathbf{R}^n$.

The column space is defined as a span, so we know it is a subspace.

Check that the null space is a subspace:

1. 0 is in Nul A because $A0 = 0$.

2. If u and v are in Nul A, then $Au = 0$ and $Av = 0$. Hence
$$A(u + v) = Au + Av = 0,$$
so $u + v$ is in Nul A.

3. If u is in Nul A, then $Au = 0$. For any scalar c, $A(cu) = cAu = 0$. So cu is in Nul A.

Column Space and Null Space
Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the column space:

$$\text{Col } A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

This is a line in $\mathbf{R}^3$.
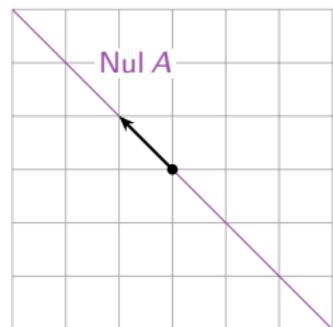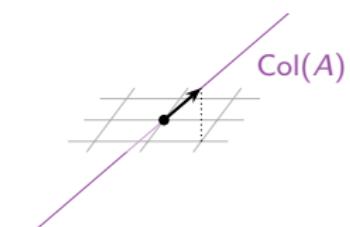
Let's compute the null space:

The reduced row echelon form of A is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives the equation $x + y = 0$, or

$\begin{aligned} x &= -y \\ y &= y \end{aligned}$ $\xrightarrow{\text{parametric vector form}}$ $\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

Hence the null space is $\text{Span}\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$, a line in $\mathbf{R}^2$.

Col(A)

Nul A

The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question
How to find vectors that span the null space?

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{array}{c} \text{if, say, } x_3 \text{ and } x_4 \\ \text{are the free} \\ \text{variables. So} \end{array} \qquad \text{Nul } A = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §2.4 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

- A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:

    Col A = the span of the columns of A

    Nul A = the solution set of $Ax = 0$.

How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we'll learn about later (eigenspaces, . . .)?

If so, then it's automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?

Sections 2.7 and 2.9

Basis, Dimension, Rank and Basis Theorems

Recall: a subspace of $\mathbf{R}^n$ is the same thing as a span, except we haven't computed a spanning set yet.

For example, Col A and Nul A for a matrix A.

There are lots of choices of spanning set for a given subspace.

Are some better than others?

Basis of a Subspace

What is the *smallest number* of vectors that are needed to span a subspace?

> **Definition**
> Let V be a subspace of $\mathbf{R}^n$. A **basis** of V is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in V such that:
>
> 1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
>
> 2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.
>
> The number of vectors in a basis is the **dimension** of V, and is written $\dim V$.

Note the big red border here

Why is a basis the smallest number of vectors needed to span?

Recall: *linearly independent* means that every time you add another vector, the span gets bigger.

Hence, if we remove any vector, the span gets *smaller*: so any smaller set can't span V.

> **Important**
> A subspace has *many different* bases, but they all have the same number of vectors.
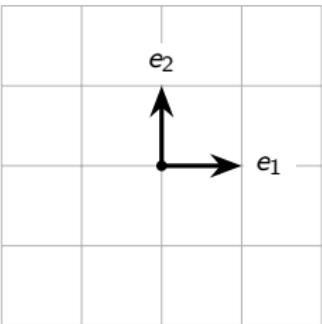
Bases of $\mathbf{R}^2$

Question

What is a basis for $\mathbf{R}^2$?

We need two vectors that *span* $\mathbf{R}^2$ and are *linearly independent*. $\{e_1, e_2\}$ is one basis.

1. They span: $\binom{a}{b} = ae_1 + be_2$.

2. They are linearly independent because they are not collinear.
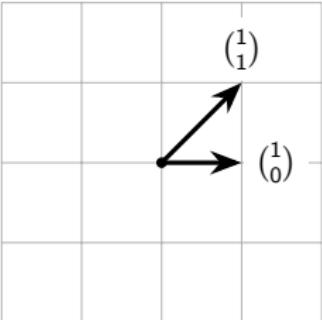
Question

What is another basis for $\mathbf{R}^2$?

Any two nonzero vectors that are not collinear. $\left\{ \binom{1}{0}, \binom{1}{1} \right\}$ is also a basis.

1. They span: $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$ has a pivot in every row.

2. They are linearly independent: $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$ has a pivot in every column.

Bases of $\mathbf{R}^n$

The unit coordinate vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

are a basis for $\mathbf{R}^n$. ⟵ The identity matrix has columns $e_1, e_2, \ldots, e_n$.

1. They span: I_n has a pivot in every row.
2. They are linearly independent: I_n has a pivot in every column.

In general: $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^n$ if and only if the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

has a pivot in every row and every column.

Sanity check: we have shown that $\dim \mathbf{R}^n = n$.

Basis of a Subspace
Example

Example

Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x + 3y + z = 0 \right\} \qquad \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}.$$

Verify that $\mathcal{B}$ is a basis for V. (So dim $V = 2$: it is a *plane*.)   [interactive]

0. In V: both vectors are in V because

$$-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.$$

1. Span: If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in V, then $y = -\frac{1}{3}(x + z)$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{x}{3} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}.$$

2. Linearly independent:

$$c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0.$$

Basis for Nul A

> **Fact**
>
> The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul A.

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{form}]{\substack{\text{parametric} \\ \text{vector}}} \quad x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow[\text{Nul } A]{\text{basis of}} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

1. The vectors span Nul A by construction (every solution to $Ax = 0$ has this form).

2. Can you see why they are linearly independent? (Look at the last two rows.)

Basis for Col A

The *pivot columns* of A always form a basis for Col A.

Warning: I mean the pivot columns of the *original* matrix A, not the row-reduced form. (Row reduction changes the column space.)

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⟷ pivot columns in rref

So a basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$

Why? See slides on linear independence.

The Basis Theorem

Basis Theorem
Let V be a subspace of dimension m. Then:

- Any m linearly independent vectors in V form a basis for V.
- Any m vectors that span V form a basis for V.

> **Upshot**
>
> If you *already* know that dim $V = m$, and you have m vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in V, then you only have to check *one* of
>
> 1. $\mathcal{B}$ is linearly independent, *or*
> 2. $\mathcal{B}$ spans V
>
> in order for $\mathcal{B}$ to be a basis.

Example: any three linearly independent vectors form a basis for $\mathbf{R}^3$.

The Rank Theorem

Recall:

- The **dimension** of a subspace V is the number of vectors in a basis for V.
- A basis for the column space of a matrix A is given by the pivot columns.
- A basis for the null space of A is given by the vectors attached to the free variables in the parametric vector form.

Definition

The **rank** of a matrix A, written rank A, is the dimension of the column space Col A. The **nullity** of A, written nullity A, is the dimension of the solution set of $Ax = 0$.

Observe:

$$\text{rank } A = \dim \text{Col } A = \text{the number of columns with pivots}$$

$$\text{nullity } A = \dim \text{Nul } A = \text{the number of free variables}$$

$$= \text{the number of columns without pivots.}$$

Rank Theorem

If A is an $m \times n$ matrix, then

$$\text{rank } A + \text{nullity } A = n = \text{the number of columns of } A.$$

In other words,   [interactive 1]   [interactive 2]

(dimension of column space) + (dimension of solution set) = (number of variables).

The Rank Theorem

Example

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of Col A · free variables

A basis for Col A is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\},$$

so rank $A = \dim \operatorname{Col} A = 2$.

Since there are two free variables x_3, x_4, the parametric vector form for the solutions to $Ax = 0$ is

$$x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{basis for Nul } A} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus nullity $A = \dim \operatorname{Nul} A = 2$.

The Rank Theorem says $2 + 2 = 4$.

Poll

Poll

True or False: If A is a 10×15 matrix and there is a basis of Col A consisting of 4 vectors, then there is a basis of Nul A consisting of 6 vectors.

False: if rank $A = 4$ then nullity $A = 15 - 4 = 11$.

Summary

- A **basis** of a subspace is a minimal set of spanning vectors.
- There are recipes for computing a basis for the column space and null space of a matrix.
- The **dimension** of a subspace is the number of vectors in any basis.
- The **basis theorem** says that if you already know that dim $V = m$, and you have m vectors in V, then you only have to check if they span *or* they're linearly independent to know they're a basis.
- The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.

Chapter 3

Linear Transformations and Matrix Algebra

Section 3.1

Matrix Transformations

Motivation

Let A be a matrix, and consider the matrix equation $b = Ax$. If we vary x, we can think of this as a *function* of x.

Many functions in real life—the *linear* transformations—come from matrices in this way.

It makes us happy when a function comes from a matrix, because then questions about the function translate into questions a matrix, which we can usually answer.
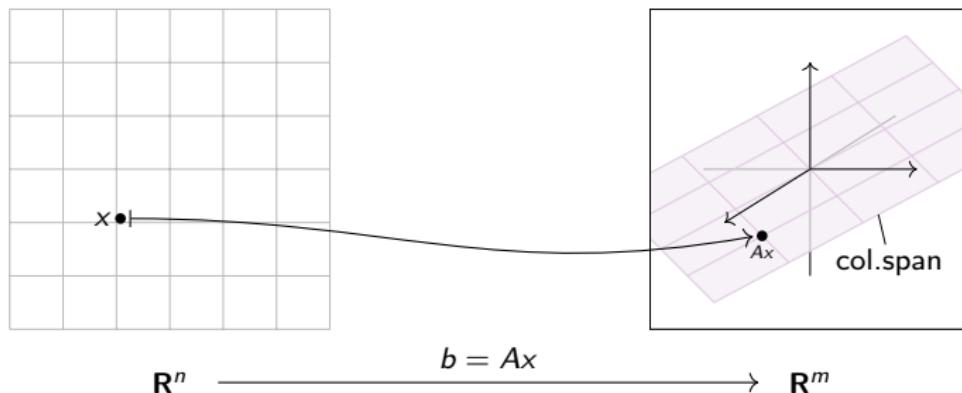
For this reason, we want to study matrices as functions.

Matrices as Functions

Change in Perspective. Let A be a matrix with m rows and n columns. Let's think about the matrix equation $b = Ax$ as a *function*.

- The independent variable (the input) is x, which is a vector in $\mathbf{R}^n$.
- The dependent variable (the output) is b, which is a vector in $\mathbf{R}^m$.

As you vary x, then $b = Ax$ also varies. The set of all possible output vectors b is the column space of A.

[interactive 1]   [interactive 2]
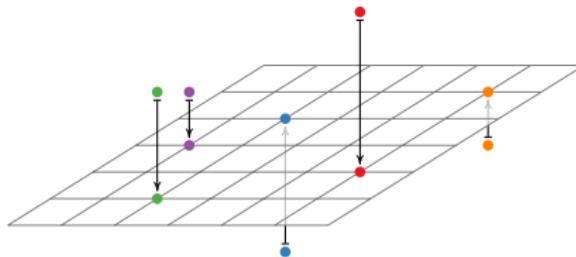
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^3$ and the output vector b is in $\mathbf{R}^3$. Then

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

This is *projection onto the xy-plane*. Picture:

[interactive]
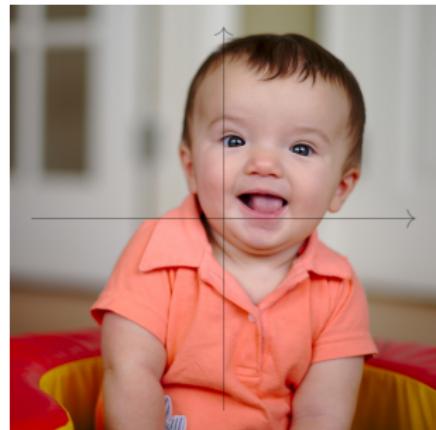
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

This is *reflection over the y-axis*. Picture:

$b = Ax$

[interactive]
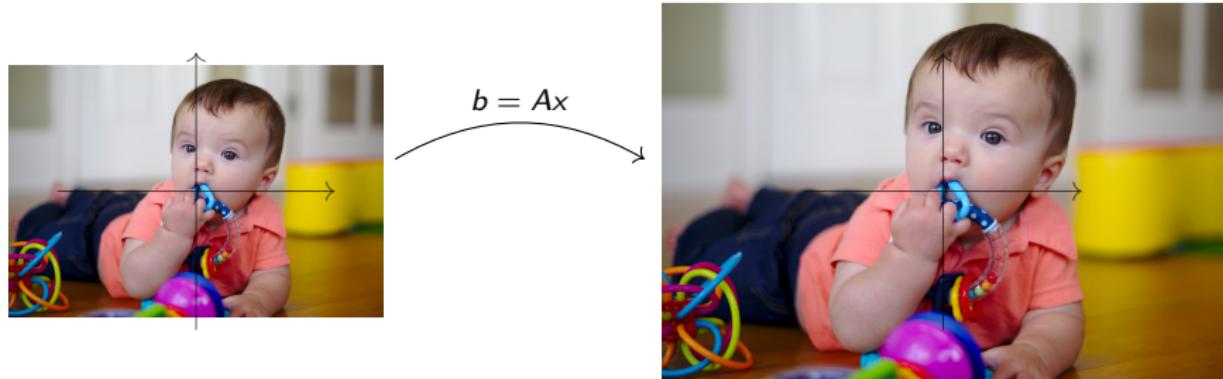
$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$.

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5 \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is *dilation (scaling) by a factor of 1.5*. Picture:
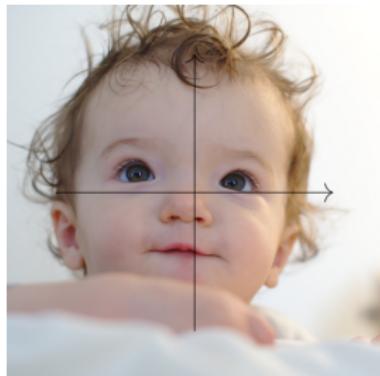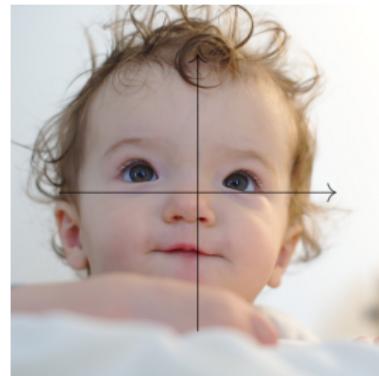
$b = Ax$

[interactive]

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$.

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

This is *the identity transformation which does nothing*. Picture:

$b = Ax$

[interactive]

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then
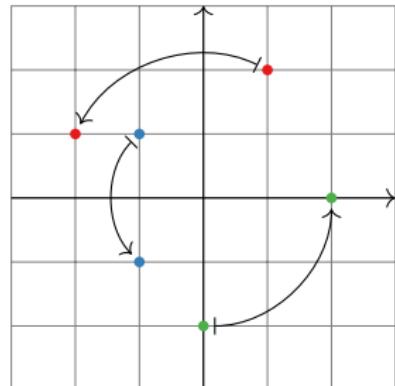
$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

What is this? Let's plug in a few points and see what happens.

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$A \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

It looks like *counterclockwise rotation by* $90°$.

Matrices as Functions
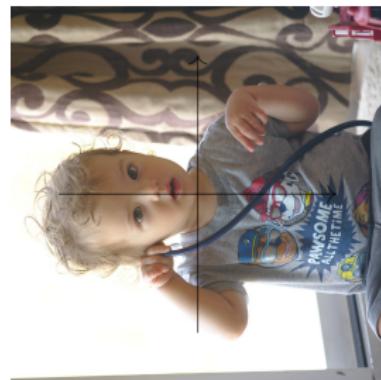
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

$b = Ax$

[interactive]

In §4.1 there are other examples of geometric transformations of $\mathbf{R}^2$ given by matrices. Please look them over.

We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let's go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?

If $\theta = 90°$, then we know $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But for general θ, it's not clear.

Our next goal is to answer this kind of question.
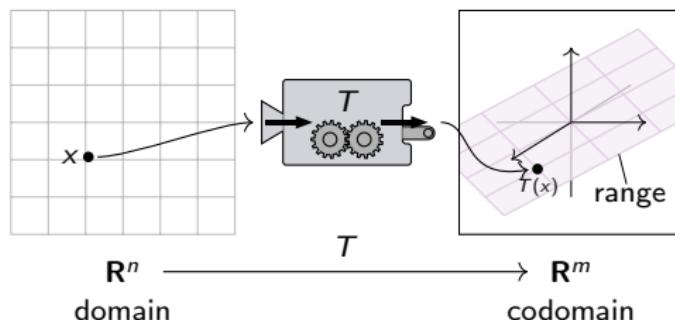
Definition
A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of T (the inputs).
- $\mathbf{R}^m$ is called the **codomain** of T (the outputs).
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of x under T.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{\, T(x) \mid x \text{ in } \mathbf{R}^n \}$ is the **range** of T.

Notation:

$$T \colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{means} \quad T \text{ is a transformation from } \mathbf{R}^n \text{ to } \mathbf{R}^m.$$

It may help to think of T as a "machine" that takes x as an input, and gives you $T(x)$ as the output.

$$\mathbf{R}^n \xrightarrow{\quad T \quad} \mathbf{R}^m$$

domain   codomain

Many of the functions you know and love have domain and codomain **R**.
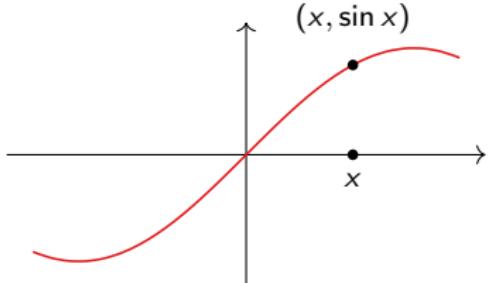
$$\sin \colon \mathbf{R} \longrightarrow \mathbf{R} \qquad \sin(x) = \begin{pmatrix} \text{the length of the opposite edge over the} \\ \text{hypotenuse of a right triangle with angle} \\ x \text{ in radians} \end{pmatrix}$$

Note how I've written down the *rule* that defines the function sin.

$$f \colon \mathbf{R} \longrightarrow \mathbf{R} \qquad f(x) = x^2$$

Note that "x^2" is sloppy (but common) notation for a function: it doesn't have a name!

You may be used to thinking of a function in terms of its graph.
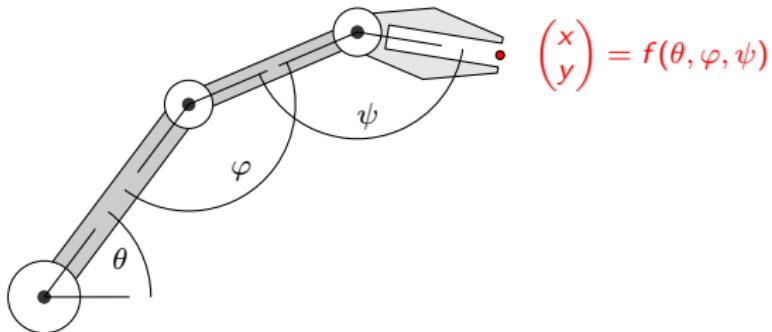
The horizontal axis is the domain, and the vertical axis is the codomain.

This is fine when the domain and codomain are **R**, but it's hard to do when they're $\mathbf{R}^2$ and $\mathbf{R}^3$! You need five dimensions to draw that graph.

Functions from Engineering

Suppose you are building a robot arm with three joints that can move its hand around a plane, as in the following picture.

Define a transformation $f \colon \mathbf{R}^3 \to \mathbf{R}^2$:

$$f(\theta, \varphi, \psi) = \text{position of the hand at joint angles } \theta, \varphi, \psi.$$

Output of f: where is the hand on the plane.

This function does not come from a matrix; belongs to the field of inverse kinematics.

Matrix Transformations

Definition
Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T \colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

In other words, T takes the vector x in $\mathbf{R}^n$ to the vector Ax in $\mathbf{R}^m$.
For example, if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $T(x) = Ax$ then

$$T \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ -32 \end{pmatrix}.$$

▶ The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
▶ The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
▶ The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the *column space* of A. It is a span of vectors in the codomain.

Your life will be much easier if you just remember these.

Matrix Transformations

Example

$$A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} \qquad T(x) = Ax \qquad T \colon \mathbf{R}^2 \to \mathbf{R}^3.$$

Domain is: $\mathbf{R}^2$.   Codomain is: $\mathbf{R}^3$.   Range is: all vectors of the form

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$
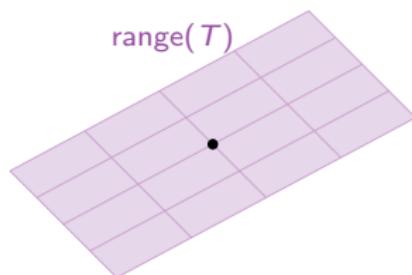
which is Col A.

range(T)

[interactive]

domain   codomain
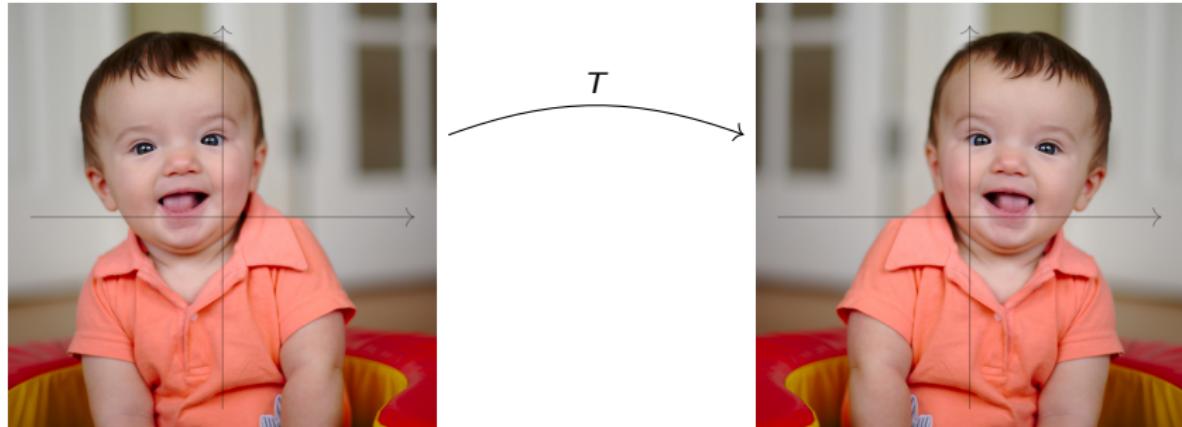
The picture of a matrix transformation is the same as the pictures we've been drawing all along. Only the language is different. Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and let} \quad T(x) = Ax,$$

so $T \colon \mathbf{R}^2 \to \mathbf{R}^2$. Then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix},$$

which is still is *reflection over the y-axis*. Picture:
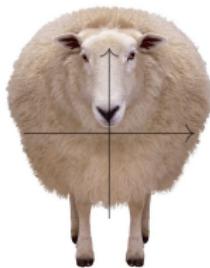
Poll

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T : \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)
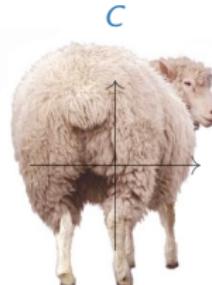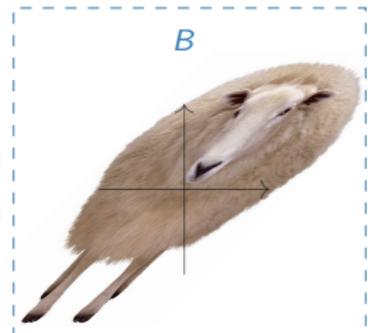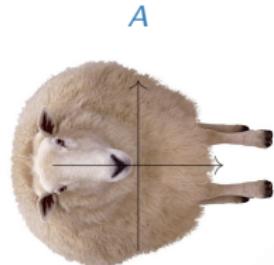
A B C

[interactive]

sheared sheep

Summary

- We can think of $b = Ax$ as a **transformation** with input x and output b.
- There are vocabulary words associated to transformations: **domain**, **codomain**, **range**.
- A transformation that comes from a matrix is called a **matrix transformation**.
- In this case, the vocabulary words mean something concrete in terms of matrices.
- We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.

Section 3.2

One-to-one and Onto Transformations

Recall: Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T \colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax.$$

- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the *column space* of A. It is a span of vectors in the codomain.

Matrix Transformations

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T \colon \mathbf{R}^2 \to \mathbf{R}^3$.

▶ If $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then $T(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$.

▶ Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?

We want to find v such that $T(v) = Av = b$. We know how to do that:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} v = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix} \xrightarrow[\text{matrix}]{\text{augmented}} \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \xrightarrow[\text{reduce}]{\text{row}} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right).$$

This gives $x = 2$ and $y = 5$, or $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ (unique). In other words,

$$T(v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T \colon \mathbf{R}^2 \to \mathbf{R}^3$.

► Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?

Translation: is there any c in $\mathbf{R}^3$ such that the solution set of $Ax = c$ has more than one vector v in it?

The solution set of $Ax = c$ is a translate of the solution set of $Ax = b$ (from before), which has one vector in it. So the solution set to $Ax = c$ has only one vector. So no!

► Find c such that there is *no* v with $T(v) = c$.

Translation: Find c such that $Ax = c$ is inconsistent.

Translation: Find c not in the column space of A (i.e., the range of T).

We could draw a picture, or notice that if $c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then our matrix equation translates into

$$x + y = 1 \qquad y = 2 \qquad x + y = 3,$$

which is obviously inconsistent.

Matrix Transformations

Note: All of these questions are questions about *the transformation T*; it still makes sense to ask them in the absence of the matrix A.

The fact that T comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T \colon \mathbf{R}^2 \to \mathbf{R}^3$ $\qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

Question: Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?

Note the question still makes sense, although T has no hope of being a matrix transformation.

By the way,

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin 0 \\ 0 \cdot 0 \\ \cos 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin \pi \\ 0 \cdot \pi \\ \cos 0 \end{pmatrix} = T\begin{pmatrix} \pi \\ 0 \end{pmatrix},$$

so the answer is yes.

Questions About Transformations

Today we will focus on two important questions one can ask about a transformation $T: \mathbf{R}^n \to \mathbf{R}^m$:

- Do there exist distinct vectors x, y in $\mathbf{R}^n$ such that $T(x) = T(y)$?

- For every vector v in $\mathbf{R}^m$, does there exist a vector x in $\mathbf{R}^n$ such that $T(x) = v$?

These are subtle because of the multiple *quantifiers* involved ("for every", "there exists").

One-to-one Transformations

Definition

A transformation $T: \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbf{R}^n$ map to different vectors in $\mathbf{R}^m$. In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at most one* solution x. Or, different inputs have different outputs. Note that *not* one-to-one means at least two different vectors in $\mathbf{R}^n$ have the same image.
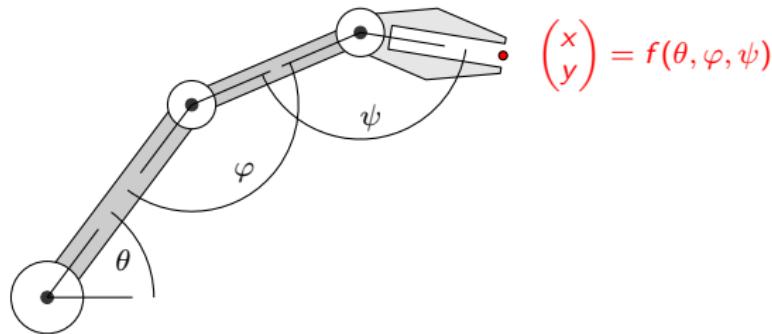
Consider the robot hand transformation from last lecture:

$$\begin{pmatrix} x \\ y \end{pmatrix} = f(\theta, \varphi, \psi)$$

Define $f : \mathbf{R}^3 \to \mathbf{R}^2$ by:

$$f(\theta, \varphi, \psi) = \text{position of the hand at joint angles } \theta, \varphi, \psi.$$

Poll

Is f one-to-one?

No: there is more than one way to move the hand to the same point.

Characterization of One-to-One Matrix Transformations

Theorem
Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is one-to-one.
- For each b in $\mathbf{R}^m$, the equation $T(x) = b$ has at most one solution.
- For each b in $\mathbf{R}^m$, the equation $Ax = b$ has a unique solution or is inconsistent.
- $Ax = 0$ has a unique solution.
- The columns of A are linearly independent.
- A has a pivot in every column.

Question
If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every column, it must have *at least as many* rows as columns: $n \leq m$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

For instance, $\mathbf{R}^3$ is "too big" to map *into* $\mathbf{R}^2$.

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T(x) = Ax,$$

so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$. Is T one-to-one?

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which has a pivot in every column. Hence T is one-to-one.

[interactive]

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad T(x) = Ax,$$

so $T \colon \mathbf{R}^3 \to \mathbf{R}^2$. Is T one-to-one? If not, find two different vectors x, y such that $T(x) = T(y)$.

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

which does not have a pivot in every column. Hence A is not one-to-one. In particular, $Ax = 0$ has nontrivial solutions. The parametric form of the solutions of $Ax = 0$ are

$$\begin{aligned} x \quad\ \ - z &= 0 \\ y + z &= 0 \end{aligned} \quad\Longrightarrow\quad \begin{aligned} x &= \ \ z \\ y &= -z. \end{aligned}$$

Taking $z = 1$ gives
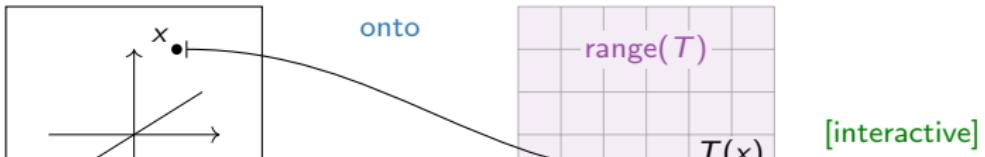
$$T\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0 = T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

[interactive]
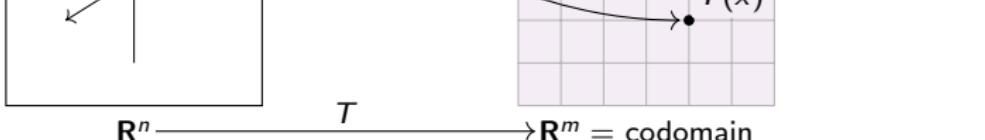
Onto Transformations

Definition

A transformation $T \colon \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbf{R}^m$ (its codomain). In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has at *least one solution*. Or, every possible output has an input. Note that *not* onto means there is some b in $\mathbf{R}^m$ which is not the image of any x in $\mathbf{R}^n$.
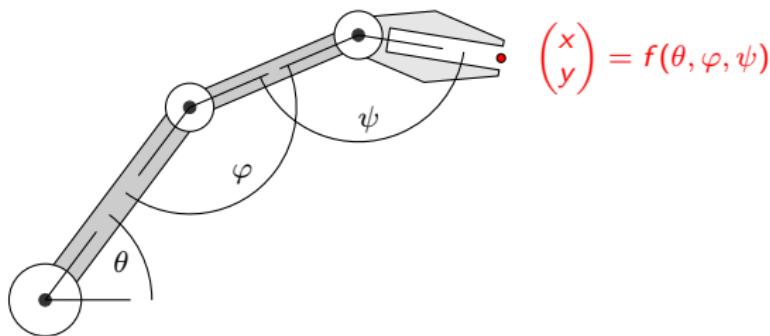
Poll

Consider the robot hand transformation again:

$$\begin{pmatrix} x \\ y \end{pmatrix} = f(\theta, \varphi, \psi)$$

Define $f \colon \mathbf{R}^3 \to \mathbf{R}^2$ by:

$$f(\theta, \varphi, \psi) = \text{position of the hand at joint angles } \theta, \varphi, \psi.$$

Poll

Is f onto?

No: it can't reach points that are far away.

Characterization of Onto Matrix Transformations

Theorem

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is onto
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is consistent for every b in $\mathbf{R}^m$
- The columns of A span $\mathbf{R}^m$
- A has a pivot in every row

Question

If $T: \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every row, it must have *at least as many* columns as rows: $m \leq n$.

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix}$$

For instance, $\mathbf{R}^2$ is "too small" to map *onto* $\mathbf{R}^3$.

Onto Transformations

Example

Define

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad T(x) = Ax,$$

so $T \colon \mathbf{R}^3 \to \mathbf{R}^2$. Is T onto?

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

which has a pivot in every row. Hence T is onto.

Note that T is *onto* but not *one-to-one*.

[interactive]

Onto Transformations

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T(x) = Ax,$$

so $T: \mathbf{R}^2 \to \mathbf{R}^3$. Is T onto? If not, find a vector v in $\mathbf{R}^3$ such that there does not exist any x in $\mathbf{R}^2$ with $T(x) = v$.

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which does not have a pivot in every row. Hence A is not onto.

In order to find a vector v not in the range, we notice that $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ a \end{pmatrix}$. In particular, the x- and z-coordinates are the same for every vector in the range, so for example, $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is not in the range.

Note that T is *one-to-one* but not *onto*.

[interactive]

Define

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \qquad T(x) = Ax,$$

so $T \colon \mathbf{R}^3 \to \mathbf{R}^2$. Is T one-to-one? Is it onto?

The reduced row echelon form of A is

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix},$$

which does not have a pivot in every row or in every column. Hence T is neither one-to-one nor onto.

[interactive]

Summary

- A transformation T is **one-to-one** if $T(x) = b$ has *at most one* solution, for every b in $\mathbf{R}^m$.
- A transformation T is **onto** if $T(x) = b$ has *at least one* solution, for every b in $\mathbf{R}^m$.
- A matrix transformation with matrix A is one-to-one if and only if the columns of A are linearly independent, if and only if A has a pivot in every column.
- A matrix transformation with matrix A is onto if and only if the columns of A span $\mathbf{R}^m$, if and only if A has a pivot in every row.
- Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.

Section 3.3

Linear Transformations
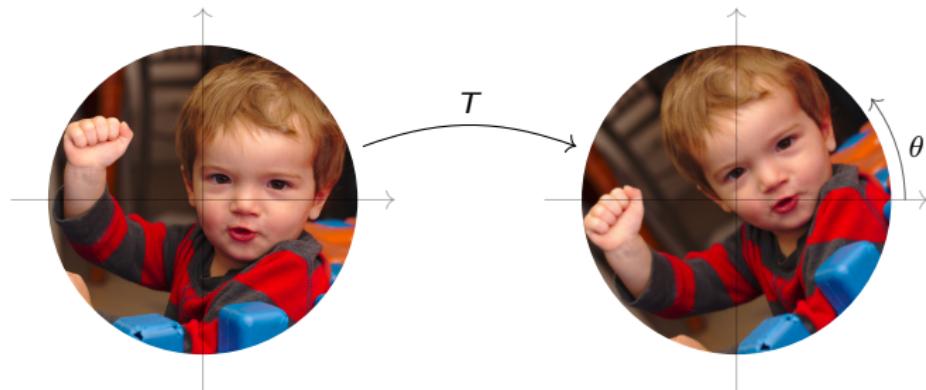
In the last two lectures we have been asking questions about transformations, and answering them in the case of matrix transformations.

However, sometimes it is not clear if a transformation is a matrix transformation or not.

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?

Today we will answer this question.

So, which transformations actually come from matrices?

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then
$$A(u + v) = Au + Av \qquad A(cv) = cAv.$$
So if $T(x) = Ax$ is a matrix transformation then,
$$T(u + v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$
Any matrix transformation has to satisfy this property. This property is so special that it has its own name.

Definition
A transformation $T \colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

In other words, T "respects" addition and scalar multiplication.

Check: if T is linear, then
$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$
for all vectors u, v and scalars c, d. More generally,
$$T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n).$$
In engineering this is called **superposition**.

Define $T: \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

$$T(u + v) = 1.5(u + v) = 1.5u + 1.5v = T(u) + T(v)$$
$$T(cv) = 1.5(cv) = c(1.5v) = c(Tv).$$

So T satisfies the two equations, hence T is linear.

Note: T is a matrix transformation!

$$T(x) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} x,$$

as we checked before.
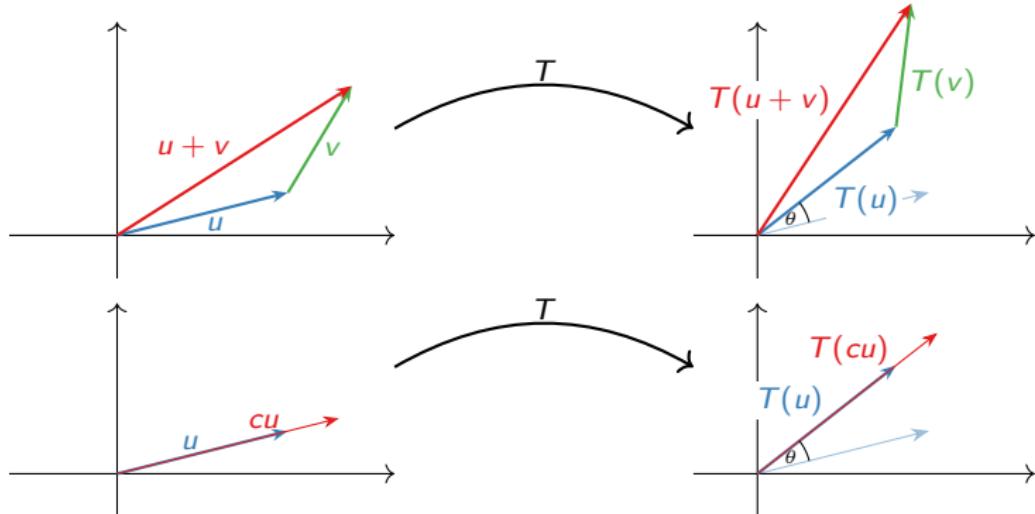
Define $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$T(x) =$ the vector x rotated counterclockwise by an angle of θ.

Is T linear? Check:

The pictures show $T(u) + T(v) = T(u + v)$ and $T(cu) = cT(u)$.

Since T satisfies the two equations, T is linear.

Is every transformation a linear transformation?

No! For instance, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$ is not linear.

Why? We have to check the two defining properties. Let's try the second:

$$T\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \sin(cx) \\ (cx)(cy) \\ \cos(cy) \end{pmatrix} \overset{?}{=} c\begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix} = cT\begin{pmatrix} x \\ y \end{pmatrix}$$

Not necessarily: if $c = 2$ and $x = \pi$, $y = \pi$, then

$$T\left(2\begin{pmatrix} \pi \\ \pi \end{pmatrix}\right) = T\begin{pmatrix} 2\pi \\ 2\pi \end{pmatrix} = \begin{pmatrix} \sin 2\pi \\ 2\pi \cdot 2\pi \\ \cos 2\pi \end{pmatrix} = \begin{pmatrix} 0 \\ 4\pi^2 \\ 1 \end{pmatrix}$$

$$2T\begin{pmatrix} \pi \\ \pi \end{pmatrix} = 2\begin{pmatrix} \sin \pi \\ \pi \cdot \pi \\ \cos \pi \end{pmatrix} = \begin{pmatrix} 0 \\ 2\pi^2 \\ -2 \end{pmatrix}.$$

So T fails the second property. Conclusion: T is *not* a matrix transformation!

(We could also have noted $T(0) \neq 0$.)

Poll

Poll

Which of the following transformations are linear?

A. $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} |x_1| \\ x_2 \end{pmatrix}$ **B.** $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$

C. $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix}$ **D.** $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 1 \\ x_1 - 2x_2 \end{pmatrix}$

A. $T \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 0 \end{pmatrix} = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} + T \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, so not linear.

B. Linear.

C. $T \left(2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \neq 2T \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so not linear.

D. $T \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0$, so not linear.

Remark: in fact, T is linear if and only if each entry of the output is a linear function of the entries of the input, with no constant terms. Check this!

The Matrix of a Linear Transformation

We will see that a *linear* transformation T is a matrix transformation: $T(x) = Ax$.

But what matrix does T come from? What is A?

Here's how to compute it.
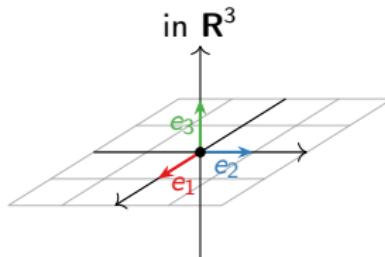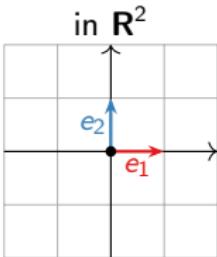
Unit Coordinate Vectors

Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

This is what $e_1, e_2, \ldots$ mean, for the rest of the class.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

in $\mathbf{R}^2$

in $\mathbf{R}^3$

Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by e_i gives you the ith column.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

Linear Transformations are Matrix Transformations

Recall: A matrix A defines a linear transformation T by $T(x) = Ax$.

Theorem

Let $T \colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.
The matrix A is called the **standard matrix** for T.

> **Take-Away**
>
> Linear transformations are the same as matrix transformations.

Dictionary

Linear transformation
$T \colon \mathbf{R}^n \to \mathbf{R}^m$ $\rightsquigarrow$ $m \times n$ matrix $A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$

$T(x) = Ax$
$T \colon \mathbf{R}^n \to \mathbf{R}^m$ $\leftrightsquigarrow$ $m \times n$ matrix A

Why is a linear transformation a matrix transformation?

Suppose for simplicity that $T \colon \mathbf{R}^3 \to \mathbf{R}^2$.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$= T\left(xe_1 + ye_2 + ze_3 \right)$$

$$= xT(e_1) + yT(e_2) + zT(e_3)$$

$$= \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Before, we defined a **dilation** transformation $T: \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$.
What is its standard matrix?

$$\left.\begin{array}{l} T(e_1) = 1.5e_1 = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \\[2em] T(e_2) = 1.5e_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \end{array}\right\} \implies A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5 \begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T(x) = x \text{ rotated counterclockwise by an angle } \theta?$$

$$T(e_1) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

$$T(e_2) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \Bigg\} \implies A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\begin{pmatrix} \theta = 90° \implies \\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \text{from before} \end{pmatrix}$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]

$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]

$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T \colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]

$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T \colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

$$
\left.
\begin{aligned}
T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\[1em]
T(e_2) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\[1em]
T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}
\end{aligned}
\right\}
\implies A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
$$

Question

Define a linear transformation $T \colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ -y - 5z \end{pmatrix}.$$

What is the standard matrix A for T?

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$\implies A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \end{pmatrix}.$$

A linear transformation is a matrix transformation, so questions about linear transformations are questions about matrices.

Question

Let $T: \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that reflects through the xy-plane and then projects onto the yz-plane. Is T one-to-one?

We have $T(x) = Ax$ for

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This does not have a pivot in the first column, so T is not one-to-one.

Summary

- **Linear transformations** are the transformations that come from matrices.
- The **unit coordinate vectors** $e_1, e_2, \ldots$ are the unit vectors in the positive direction along the coordinate axes.
- You compute the columns of the matrix for a linear transformation by plugging in the unit coordinate vectors.
- This is useful when the transformation is specified geometrically, in terms of a formula, or any other way that isn't as a matrix transformation.

Section 3.4

Matrix Multiplication

Motivation

Recall: we can turn any system of linear equations into a matrix equation

$$Ax = b.$$

This notation is suggestive. Can we solve the equation by "dividing by A"?

$$x \overset{??}{=} \frac{b}{A}$$

Answer: Sometimes, but you have to know what you're doing.

Today we'll study *matrix algebra*: adding and multiplying matrices.

These are not so hard to do. The important thing to understand today is the relationship between *matrix multiplication* and *composition of transformations*.

More Notation for Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the ij**th entry** of the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \quad i\text{th row}$$

jth column

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$A$$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \rightsquigarrow$$

flip

$$A^T$$
$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$$

Matrix Multiplication

Beware: matrix multiplication is more subtle than addition and scalar multiplication.

must be equal

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2 \ldots, v_p$:

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

The equality is a definition

$$AB \stackrel{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B. Note the sizes of the product!

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$$

The Row-Column Rule for Matrix Multiplication

Recall: A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} -\,r_1\,- \\ \vdots \\ -\,r_m\,- \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix}.$$

It follows that

$$AB = \begin{pmatrix} -\,r_1\,- \\ \vdots \\ -\,r_m\,- \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1 c_1 & r_1 c_2 & \cdots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m c_2 & \cdots & r_m c_p \end{pmatrix}$$

The Row-Column Rule for Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:
$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This is how everybody on the planet actually computes AB. Diagram ($AB = C$):

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

ith row

jth column

ij entry

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$

Composition of Transformations
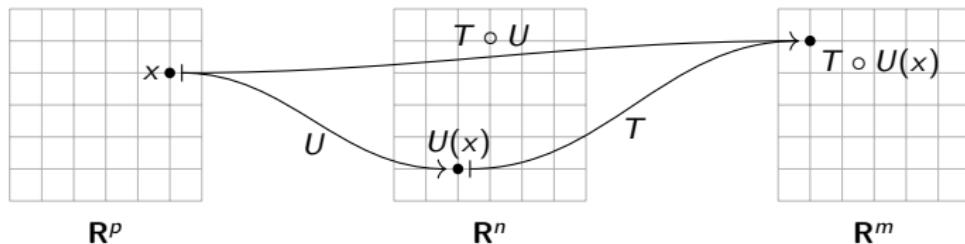
Why is this the correct definition of matrix multiplication?

Definition
Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be transformations. The **composition** is the transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \quad \text{defined by} \quad T \circ U(x) = T(U(x)).$$

This makes sense because $U(x)$ (the output of U) is in $\mathbf{R}^n$, which is the domain of T (the inputs of T).   [interactive]

Fact: If T and U are linear then so is $T \circ U$.

Guess: If A is the matrix for T, and B is the matrix for U, what is the matrix for $T \circ U$?

Composition of Linear Transformations

Let $T: \mathbf{R}^n \to \mathbf{R}^m$ and $U: \mathbf{R}^p \to \mathbf{R}^n$ be *linear* transformations. Let A and B be their matrices:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix} \qquad B = \begin{pmatrix} | & | & & | \\ U(e_1) & U(e_2) & \cdots & U(e_p) \\ | & | & & | \end{pmatrix}$$

Question

$U(e_1) = Be_1$ is the first column of B

What is the matrix for $T \circ U$?

the first column of AB is $A(Be_1)$

We find the matrix for $T \circ U$ by plugging in the unit coordinate vectors:

$$T \circ U(e_1) = T(U(e_1)) \stackrel{\shortmid}{=} T(Be_1) = A(Be_1) \stackrel{\shortmid}{=} (AB)e_1.$$

For any other i, the same works:

$$T \circ U(e_i) = T(U(e_i)) = T(Be_i) = A(Be_i) = (AB)e_i.$$

This says that the ith column of the matrix for $T \circ U$ is the ith column of AB.

> The matrix of the composition is the product of the matrices!

Addition and Scalar Multiplication for Linear Transformations
Remark

We can also add and scalar multiply linear transformations:

$T, U \colon \mathbf{R}^n \to \mathbf{R}^m \quad \rightsquigarrow \quad T + U \colon \mathbf{R}^n \to \mathbf{R}^m \qquad (T + U)(x) = T(x) + U(x).$

In other words, add transformations "pointwise".

$T \colon \mathbf{R}^n \to \mathbf{R}^m \qquad c \text{ in } \mathbf{R} \quad \rightsquigarrow \quad cT \colon \mathbf{R}^n \to \mathbf{R}^m \qquad (cT)(x) = c \cdot T(x).$

In other words, scalar-multiply a transformation "pointwise".

The next slide describes these operations in terms of matrix algebra.

Addition and Scalar Multiplication for Matrices

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size*.

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} c a_{11} & c a_{12} & c a_{13} \\ c a_{21} & c a_{22} & c a_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$A + B = B + A \qquad (A + B) + C = A + (B + C)$$
$$c(A + B) = cA + cB \qquad (c + d)A = cA + dA$$
$$(cd)A = c(dA) \qquad A + 0 = A$$

If linear transformations T and U have matrices A and B, respectively:

- $T + U$ has matrix $A + B$.
- cT has matrix cA.

Let $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ and $U\colon \mathbf{R}^2 \to \mathbf{R}^3$ be the matrix transformations

$$T(x) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x \qquad U(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} x.$$

Then the matrix for $T \circ U$ is
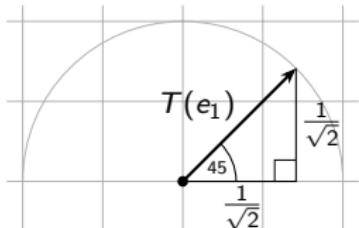
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

[interactive]
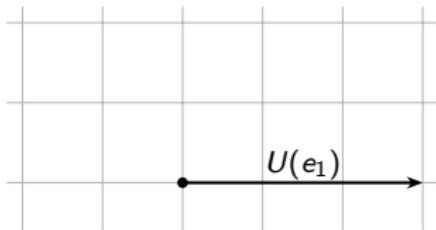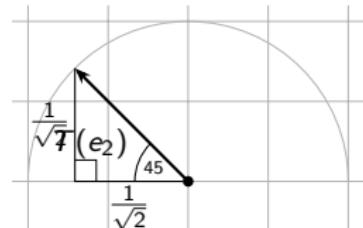
Composition of Linear Transformations
Another Example

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be rotation by $45°$, and let $U\colon \mathbf{R}^2 \to \mathbf{R}^2$ scale the x-coordinate by 1.5. Let's compute their standard matrices A and B:
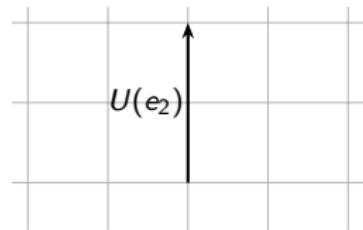
$$T(e_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T(e_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$U(e_1) = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix}$$

$$U(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\implies \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1.5 & 0 \\ 0 & 1 \end{pmatrix}$$

So the matrix C for $T \circ U$ is

$$C = AB = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1.5 & -1 \\ 1.5 & 1 \end{pmatrix}.$$

Check: [interactive: e_1]   [interactive: e_2]

$$T \circ U(e_1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$$

$$T \circ U(e_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\implies C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1.5 & -1 \\ 1.5 & 1 \end{pmatrix} \quad \checkmark$$

Let $T \colon \mathbf{R}^3 \to \mathbf{R}^3$ be projection onto the yz-plane, and let $U \colon \mathbf{R}^3 \to \mathbf{R}^3$ be reflection over the xy-plane. Let's compute their standard matrices A and B:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So the matrix C for $T \circ U$ is

$$C = AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Check: we did this last time ✔

[interactive: e_1]   [interactive: e_2]   [interactive: e_3]

Poll

Poll

Do there exist *nonzero* matrices A and B with $AB = 0$?

Yes! Here's an example:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Properties of Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$A(BC) = (AB)C \qquad A(B + C) = (AB + AC)$$
$$(B + C)A = BA + CA \qquad c(AB) = (cA)B$$
$$c(AB) = A(cB) \qquad I_m A = A$$
$$AI_n = A$$

Most of these are easy to verify.

Associativity is $A(BC) = (AB)C$. It is a pain to verify using the row-column rule! Much easier: use associativity of linear transformations:

$$S \circ (T \circ U) = (S \circ T) \circ U.$$

This is a good example of an instance where having a conceptual viewpoint saves you a lot of work.

Recommended: Try to verify all of them on your own.

Warnings!

► AB is usually not equal to BA.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

In fact, AB may be defined when BA is not.

► $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

► $AB = 0$ does not imply $A = 0$ or $B = 0$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Powers of a Matrix

Suppose A is a *square* matrix.

Then $A \cdot A$ makes sense, and has the same size.

Then $A \cdot (A \cdot A)$ also makes sense and has the same size.

Definition
Let n be a positive whole number and let A be a square matrix. The nth **power** of A is the product
$$A^n = \underbrace{A \cdot A \cdots A}_{n \text{ times}}$$

Example
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\cdots \quad A^n = \begin{pmatrix} 1 & n-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Summary

▶ The product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix. I showed you two ways of computing the product.

▶ Composition of linear transformations corresponds to multiplication of matrices.

▶ You have to be careful when multiplying matrices together, because things like commutativity and cancellation fail.

▶ You can take powers of square matrices.

Section 3.5 and 3.6

Matrix Inverses and the Invertible Matrix Theorem

The Definition of Inverse
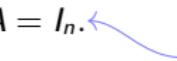
Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that $ab = 1$. We define the inverse of a matrix in almost the same way.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

In this case, B is the **inverse** of A, and is written A^{-1}.

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

I claim $B = A^{-1}$. Check:

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

✔

Poll

Poll

Do there exist two matrices A and B such that AB is the identity, but BA is not? If so, find an example. (Both products have to make sense.)

Yes, for instance:
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

However

If A and B are *square* matrices, then

$$AB = I_n \quad \text{if and only if} \quad BA = I_n.$$

So in this case you only have to check one.

Theorem

If A is invertible, then $Ax = b$ has exactly one solution for every b, namely:

$$x = A^{-1}b.$$

Why? Divide by A!

$$Ax = b \rightsquigarrow A^{-1}(Ax) = A^{-1}b \rightsquigarrow (A^{-1}A)x = A^{-1}b$$
$$\rightsquigarrow (I_n x) = A^{-1}b \rightsquigarrow x = A^{-1}b.$$

$I_n x = x$ for every x

Important

If A is invertible and you know its inverse, then the easiest way to solve $Ax = b$ is by "dividing by A":

$$x = A^{-1}b.$$

This is very convenient when you have to vary b!

Example

Solve the system

$$
\begin{aligned}
2x_1 + 3x_2 + 2x_3 &= 1 \\
x_1 + 3x_3 &= 1 \\
2x_1 + 2x_2 + 3x_3 &= 1
\end{aligned}
\qquad \text{using} \qquad
\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1}
=
\begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.
$$

Answer:
$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
=
\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1}
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
=
\begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
=
\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.
$$

The advantage of using inverses is it doesn't matter what's on the right-hand side of the $=$:

$$
\begin{cases}
2x_1 + 3x_2 + 2x_3 = b_1 \\
x_1 + 3x_3 = b_2 \\
2x_1 + 2x_2 + 3x_3 = b_3
\end{cases}
\implies
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
=
\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1}
\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
$$

$$
=
\begin{pmatrix} -6b_1 - 5b_2 + 9b_3 \\ 3b_1 + 2b_2 - 4b_3 \\ 2b_1 + 2b_2 - 3b_3 \end{pmatrix}.
$$

Some Facts

Say A and B are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible and its inverse is $(AB)^{-1} = \cancel{A^{-1}B^{-1}}$ $B^{-1}A^{-1}$.

   Why? $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$.
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

   Why? $A^T(A^{-1})^T = (A^{-1}A)^T = I_n^T = I_n$.

Question: If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC?

   i. $A^{-1}B^{-1}C^{-1}$ ii. $B^{-1}A^{-1}C^{-1}$ $\boxed{\text{iii. } C^{-1}B^{-1}A^{-1}}$ iv. $C^{-1}A^{-1}B^{-1}$

Check:
$$(ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} = A(BB^{-1})A^{-1}$$
$$= AA^{-1} = I_n.$$

In general, a product of invertible matrices is invertible, and the inverse is the product of the inverses, in the *reverse order*.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Facts:

1. If $\det(A) \neq 0$, then A is invertible and $\boxed{A^{-1} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$.

2. If $\det(A) = 0$, then A is not invertible.

Why 1?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So we get the identity by dividing by $ad - bc$.

Example

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2 \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}.$$

Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

1. Row reduce the augmented matrix $(\, A \mid I_n \,)$.
2. If the result has the form $(\, I_n \mid B \,)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

[interactive]

Computing A^{-1}
Example

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \quad \xrightarrow{\begin{array}{c} R_3 = R_3 + 3R_2 \end{array}} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\begin{array}{c} R_1 = R_1 - 2R_3 \\ R_2 = R_2 - R_3 \end{array} \xrightarrow{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$R_3 = R_3 \div 2 \xrightarrow{} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

So $\left(\begin{array}{ccc} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{array}\right)^{-1} = \left(\begin{array}{ccc} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{array}\right).$

Check: $\left(\begin{array}{ccc} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{array}\right) \left(\begin{array}{ccc} 1 & -6 & -2 \\ 0 & -2 & -1 \\ 0 & 3/2 & 1/2 \end{array}\right) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$ ✔

Why Does This Work?

We can think of the algorithm as simultaneously solving the equations

$$Ax_1 = e_1 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_2 = e_2 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_3 = e_3 : \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

Now note $A^{-1}e_i = A^{-1}(Ax_i) = x_i$, and x_i is the ith column in the augmented part. Also $A^{-1}e_i$ is the ith column of A^{-1}.

Definition

A transformation $T \colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U \colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbf{R}^n$. In this case we say U is the **inverse** of T, and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U, and likewise U "undoes" T.

Fact
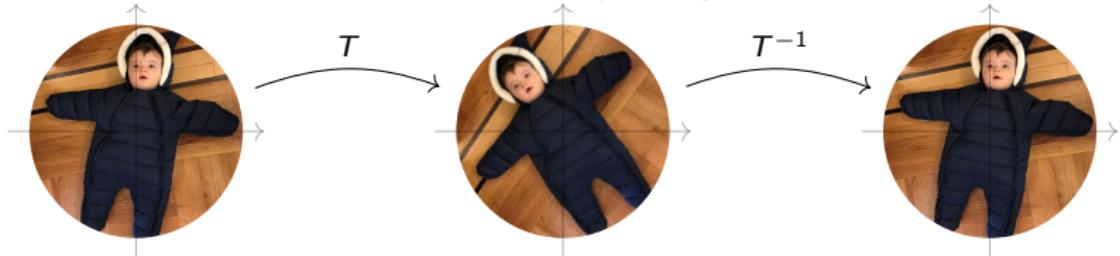
A transformation T is invertible if and only if it is both one-to-one and onto.

If T is one-to-one and onto, this means for every y in $\mathbf{R}^n$, there is a unique x in $\mathbf{R}^n$ such that $T(x) = y$. Then $T^{-1}(y) = x$.

Invertible Transformations

Examples
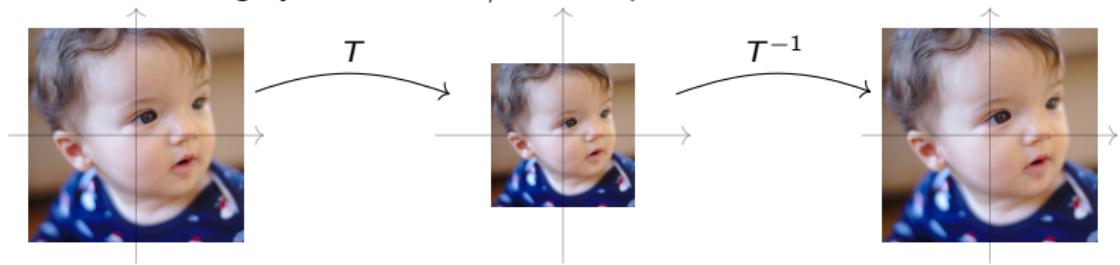
Let $T =$ counterclockwise rotation in the plane by $45°$. What is T^{-1}?

T^{-1} is *clockwise* rotation by $45°$.   [interactive: $T^{-1} \circ T$] [interactive: $T \circ T^{-1}$]

Let $T =$ shrinking by a factor of $2/3$ in the plane. What is T^{-1}?

T^{-1} is *stretching* by $3/2$.   [interactive: $T^{-1} \circ T$] [interactive: $T \circ T^{-1}$]

Let $T =$ projection onto the x-axis. What is T^{-1}? It is not invertible: you can't undo it.

If $T: \mathbf{R}^n \to \mathbf{R}^n$ is an invertible *linear* transformation with matrix A, then what is the matrix for T^{-1}?

Let B be the matrix for T^{-1}. We know $T \circ T^{-1}$ has matrix AB, so for all x,

$$ABx = T \circ T^{-1}(x) = x.$$

Hence $AB = I_n$, so $B = A^{-1}$.

> **Fact**
>
> If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

Let T = counterclockwise rotation in the plane by $45°$. Its matrix is

$$A = \begin{pmatrix} \cos(45°) & -\sin(45°) \\ \sin(45°) & \cos(45°) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then T^{-1} = counterclockwise rotation by $-45°$. Its matrix is

$$B = \begin{pmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Check: $\qquad AB = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✔

Let T = shrinking by a factor of $2/3$ in the plane. Its matrix is

$$A = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix}$$

Then T^{-1} = stretching by $3/2$. Its matrix is

$$B = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$

Check: $\qquad AB = \begin{pmatrix} 2/3 & 0 \\ 0 & 2/3 \end{pmatrix} \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✔

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T \colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is the identity matrix I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial solution.
6. $\text{Nul}(A) = \{0\}$.
7. $\text{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\text{Col } A = \mathbf{R}^n$.
15. $\dim \text{Col } A = n$.
16. $\text{rank } A = n$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

you really have to know these

The Invertible Matrix Theorem
Summary

There are two kinds of *square* matrices:

1. invertible (non-singular), and
2. non-invertible (singular).

For invertible matrices, all statements of the Invertible Matrix Theorem are true.

For non-invertible matrices, all statements of the Invertible Matrix Theorem are false.

Strong recommendation: If you want to understand invertible matrices, go through all of the conditions of the IMT and try to figure out on your own (or at least with help from the book) why they're all equivalent.

You know enough at this point to be able to reduce all of the statements to assertions about the pivots of a square matrix.

Question: Is this matrix invertible?

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -2 & -4 & 1 \end{pmatrix}$$

The second column is a multiple of the first, so the columns are linearly dependent.

A does not satisfy condition (8) of the IMT, so it is not invertible.

Problem: Let A be a 3×3 matrix such that

$$A \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

Show that the rank of A is at most 2.

If we set

$$b = A \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix},$$

then $Ax = b$ has multiple solutions, so it does not satisfy condition (12) of the IMT.

Hence it also does not satisfy condition (16), so the rank is not 3.

In any case the rank is at most 3, so it must be less than 3.

Summary

- The **inverse** of a square matrix A is a matrix A^{-1} such that $AA^{-1} = I_n$ (equivalently, $A^{-1}A = I_n$).
- If A is invertible, then you can solve $Ax = b$ by "dividing by A": $b = A^{-1}x$. There is a unique solution $x = A^{-1}b$ for every x.
- You compute A^{-1} (and whether A is invertible) by row reducing $\left(A \mid I_n \right)$. There's a trick for computing the inverse of a 2×2 matrix in terms of determinants.
- A linear transformation T is invertible if and only if its matrix A is invertible, in which case A^{-1} is the matrix for T^{-1}.
- The Invertible Matrix theorem is a list of a zillion equivalent conditions for invertibility that you have to learn (and should understand, since it's well within what we've covered in class so far).

Chapter 4

Determinants

Section 4.1

Determinants: Definition

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
  We've said most of what we'll say about this topic now.
- Solve the matrix equation $Ax = \lambda x$ (eigenvalue problem)
  We are now aiming at this.
- Almost solve the equation $Ax = b$
  This will happen later.

The next topic is *determinants*.

This is a completely magical function that takes a square matrix and gives you a number.

It is a very complicated function—the formula for the determinant of a 10×10 matrix has $3,628,800$ summands—so instead of writing down the formula, we'll give other ways to compute it.

Today is mostly about the *theory* of the determinant; in the next lecture we will focus on *computation*.

A Definition of Determinant

Definition

The **determinant** is a function ⟶ determinants are only for square matrices!

$$\det : \{n \times n \text{ matrices}\} \longrightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by c, the determinant is multiplied by c.
3. If you swap two rows of a matrix, the determinant is multiplied by -1.
4. $\det(I_n) = 1$.

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{\;R_1 \longleftrightarrow R_2\;} \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \qquad \det = 7 \;\checkmark$$

$$\xrightarrow{\;R_2 = R_2 - 2R_1\;} \begin{pmatrix} 1 & 4 \\ 0 & -7 \end{pmatrix} \qquad \det = -7$$

$$\xrightarrow{\;R_2 = R_2 \div -7\;} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \qquad \det = 1$$

$$\xrightarrow{\;R_1 = R_1 - 4R_2\;} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \det = 1$$

A Definition of Determinant

Definition
The **determinant** is a function — determinants are only for square matrices!

$$\det\colon \{n \times n \text{ matrices}\} \longrightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by c, the determinant is multiplied by c.
3. If you swap two rows of a matrix, the determinant is multiplied by -1.
4. $\det(I_n) = 1$.

> This is a *definition* because it tells you how to compute the determinant: row reduce!

It's not at all obvious that you get the same determinant if you row reduce in two different ways, but this is magically true!

Special Cases

Special Case 1

If A has a zero row, then $\det(A) = 0$.

Why?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 = -R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

The determinant of the second matrix is negative the determinant of the first (property 3), so

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}.$$

This implies the determinant is zero.

Special Cases

Special Case 2

If A is upper-triangular, then the determinant is the product of the diagonal entries:

$$\det \begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} = abc.$$

Upper-triangular means the only nonzero entries are on or above the diagonal.

Why?

▶ If one of the diagonal entries is zero, then the matrix has fewer than n pivots, so the RREF has a row of zeros. (Row operations don't change whether the determinant is zero.)

▶ Otherwise,

$$\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} \xrightarrow[]{\substack{\text{scale by} \\ a^{-1}, b^{-1}, c^{-1}}} \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[]{\substack{\text{row} \\ \text{replacements}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det = abc \xleftarrow{} \det = 1 \xleftarrow{} \det = 1$$

Theorem

Let A be a square matrix. Suppose you do some number of row operations on A to get a matrix B in row echelon form. Then

$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})},$$

where r is the number of row swaps.

Why? Since B is in REF, it is upper-triangular, so its determinant is the product of its diagonal entries. You changed the determinant by $(-1)^r$ and the product of the scaling factors when going from A to B.

Remark

This is generally the fastest way to compute a determinant of a large matrix, either by hand or by computer.

Row reduction is $O(n^3)$; cofactor expansion (next time) is $O(n!) \sim O(n^n \sqrt{n})$.

This is important in real life, when you're usually working with matrices with a gazillion columns.

Computing Determinants

Example

$$
\begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix}
\qquad
\begin{array}{c} R_1 \longleftrightarrow R_2 \\ \rightsquigarrow \end{array}
\qquad
\begin{pmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix}
\qquad r = 1
$$

$$
\begin{array}{c} R_1 = R_1 \div 2 \\ \rightsquigarrow \end{array}
\qquad
\begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix}
\qquad
\begin{array}{l} r = 1 \\ \text{scaling factors} = \tfrac{1}{2} \end{array}
$$

$$
\begin{array}{c} R_3 = R_3 - 3R_1 \\ \rightsquigarrow \end{array}
\qquad
\begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & -10 \end{pmatrix}
\qquad
\begin{array}{l} r = 1 \\ \text{scaling factors} = \tfrac{1}{2} \end{array}
$$

$$
\begin{array}{c} R_2 \longleftrightarrow R_3 \\ \rightsquigarrow \end{array}
\qquad
\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & -7 & -4 \end{pmatrix}
\qquad
\begin{array}{l} r = 2 \\ \text{scaling factors} = \tfrac{1}{2} \end{array}
$$

$$
\begin{array}{c} R_3 = R_3 + 7R_2 \\ \rightsquigarrow \end{array}
\qquad
\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{pmatrix}
\qquad
\begin{array}{l} r = 2 \\ \text{scaling factors} = \tfrac{1}{2} \end{array}
$$

$$
\implies \det \begin{pmatrix} 0 & -7 & -4 \\ 1 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} = (-1)^2 \frac{1 \cdot 1 \cdot -74}{1/2} = -148.
$$

Let's compute the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a general 2 × 2 matrix.

▶ If $a = 0$, then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc.$$

▶ Otherwise,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ 0 & d - c \cdot b/a \end{pmatrix}$$

$$= a \cdot 1 \cdot (d - bc/a) = ad - bc.$$

In both cases, the determinant magically turns out to be

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Poll

Poll

True or false:
- (a) Row operations can change the determinant of a matrix.
- (b) Row operations can change whether the determinant of a matrix is equal to zero.

(a) **True:** scaling and row swaps change the determinant by a nonzero number and by -1, respectively.

(b) **False:** all row operations multiply the determinant by a *nonzero* number.

Theorem
A square matrix A is invertible if and only if $\det(A)$ is nonzero.

Why?

- If A is invertible, then its reduced row echelon form is the identity matrix, which has determinant equal to 1.
- If A is not invertible, then its reduced row echelon form has a zero row, hence has zero determinant.
- Doing row operations doesn't change whether the determinant is zero.

Determinants and Products

Theorem

If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Why? If B is invertible, we can define

$$f(A) = \frac{\det(AB)}{\det(B)}.$$

Note $f(I_n) = \det(I_n B)/\det(B) = 1$. Check that f satisfies the same properties as det with respect to row operations. So

$$\det(A) = f(A) = \frac{\det(AB)}{\det(B)} \implies \det(AB) = \det(A)\det(B).$$

What about if B is not invertible?

Theorem

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Why? $I_n = AB \implies 1 = \det(I_n) = \det(AB) = \det(A)\det(B).$
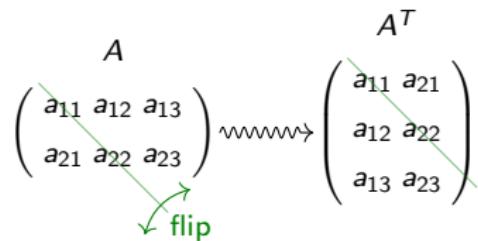
Recall: The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.

$$A$$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \rightsquigarrow \quad \begin{matrix} A^T \\ \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} \end{matrix}$$

flip

Determinants and Transposes

Theorem

If A is a square matrix, then

$$\det(A) = \det(A^T),$$

where A^T is the transpose of A.

Example: $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$

As a consequence, det behaves the same way with respect to *column* operations as row operations.

Corollary ←——— an immediate consequence of a theorem

If A has a zero column, then $\det(A) = 0$.

Corollary

The determinant of a *lower*-triangular matrix is the product of the diagonal entries.

(The transpose of a lower-triangular matrix is upper-triangular.)

Section 4.3

Determinants and Volumes
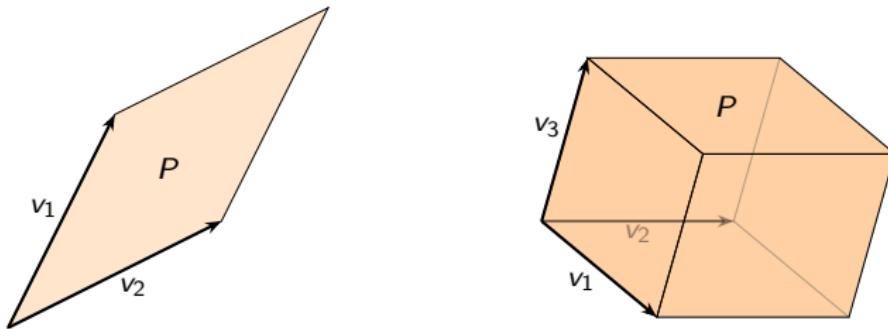
Determinants and Volumes

Now we discuss a completely different description of (the absolute value of) the determinant, in terms of volumes.

This is a crucial component of the change-of-variables formula in multivariable calculus.

The columns $v_1, v_2, \ldots, v_n$ of an $n \times n$ matrix A give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.

Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

Sanity check: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

Why is the theorem true? You only have to check that the volume behaves the same way under row operations as $|\det|$ does.

Note that the volume of the unit cube (the parallelepiped defined by the identity matrix) is 1.

Determinants and Volumes
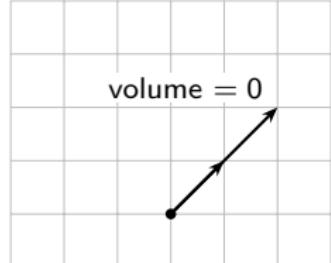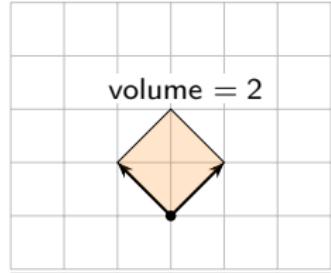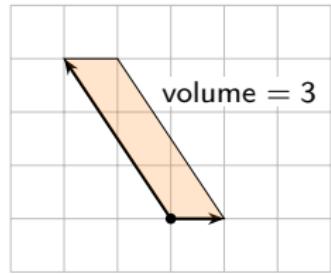
Examples in $\mathbf{R}^2$

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$

$$\det \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -2$$

(Should the volume really be -2?)

$$\det \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 0$$

volume = 3

volume = 2

volume = 0

Determinants and Volumes

Theorem
Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then
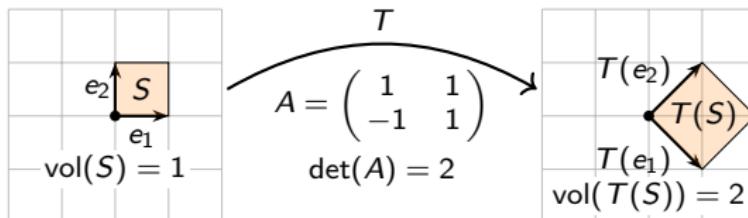
$$\text{(volume of } P) = |\det(A)|.$$

This is even true for curvy shapes, in the following sense.

Theorem
Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$\text{(volume of } T(S)) = |\det(A)| \text{ (volume of } S).$$

If S is the unit cube, then $T(S)$ is the parallelepiped defined by the columns of A, since the columns of A are $T(e_1), T(e_2), \ldots, T(e_n)$. In this case, the second theorem is the same as the first.
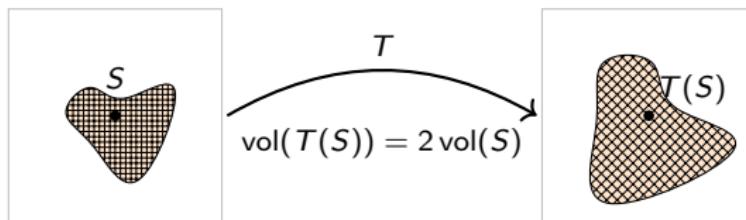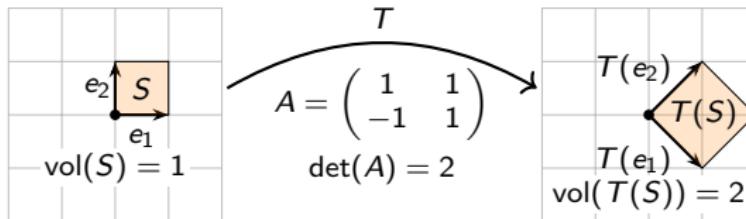
Determinants and Volumes

Theorem

Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)| (\text{volume of } S).$$

For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!

Theorem
Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)| \, (\text{volume of } S).$$

Example: Let S be the unit disk in $\mathbf{R}^2$, and let $T(x) = Ax$ for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$
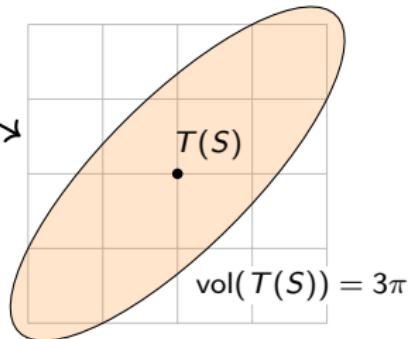
Note that $\det(A) = 3$.

Summary

Magical Properties of the Determinant

1. There is one and only one function $\det \colon \{\text{square matrices}\} \to \mathbf{R}$ satisfying the properties (1)–(4) on the second slide.

2. A is invertible if and only if $\det(A) \neq 0$.

3. The determinant of an upper- or lower-triangular matrix is the product of the diagonal entries.

4. If we row reduce A to row echelon form B using r swaps, then

$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})}.$$

5. $\det(AB) = \det(A)\det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$.

6. $\det(A) = \det(A^T)$.

7. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.

8. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)

you really have to know these

Section 4.2

Cofactor Expansions

Last time: we learned. . .

- ▶ . . . the definition of the determinant.
- ▶ . . . to compute the determinant using row reduction.
- ▶ . . . all sorts of magical properties of the determinant, like
  - ▶ $\det(AB) = \det(A)\det(B)$
  - ▶ the determinant computes volumes
  - ▶ nonzero determinants characterize invertability
  - ▶ etc.

Today: we will learn. . .

- ▶ Special formulas for 2×2 and 3×3 matrices.
- ▶ How to compute determinants using *cofactor expansions*.
- ▶ How to compute inverses using determinants.

We already have a formula in the 2×2 case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Here's the formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{matrix} a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} \\ -\ a_{13}\,a_{22}\,a_{31} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} \end{matrix}$$

How on earth do you remember this? Draw a bigger matrix, repeating the first two columns to the right:

$$+ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

Then add the products of the downward diagonals, and subtract the product of the upward diagonals. For example,

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = \begin{vmatrix} 5 & 1 & 0 & 5 & 1 \\ -1 & 3 & 2 & -1 & 3 \\ 4 & 0 & -1 & 4 & 0 \end{vmatrix} = -15 + 8 + 0 - 0 - 0 - 1 = -8$$

Cofactor Expansions

When $n \geq 4$, the determinant isn't just a sum of products of diagonals. The formula is *recursive*: you compute a larger determinant in terms of smaller ones.

First some notation. Let A be an $n \times n$ matrix.

$A_{ij} = ij$th **minor** of A

$\quad = (n-1) \times (n-1)$ matrix you get by deleting the ith row and jth column

$C_{ij} = (-1)^{i+j} \det A_{ij}$

$\quad = ij$th **cofactor** of A

The signs of the cofactors follow a checkerboard pattern:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \pm \text{ in the } ij \text{ entry is the sign of } C_{ij}$$

Theorem

The determinant of an $n \times n$ matrix A is

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

This formula is called **cofactor expansion along the first row**.

This is the beginning of the recursion.

$$\det(\, a_{11} \,) = a_{11}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The minors are:

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} \end{pmatrix} \qquad A_{12} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{21} \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12} \end{pmatrix} \qquad A_{22} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \end{pmatrix}$$

The cofactors are

$$C_{11} = + \det A_{11} = a_{22} \qquad\qquad C_{12} = - \det A_{12} = -a_{21}$$

$$C_{21} = - \det A_{21} = -a_{12} \qquad\qquad C_{22} = + \det A_{22} = a_{11}$$

The determinant is

$$\det A = a_{11} C_{11} + a_{12} C_{12} = a_{11} a_{22} - a_{12} a_{21}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The top row minors and cofactors are:

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \qquad C_{11} = + \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \qquad C_{12} = - \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad C_{13} = + \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The determinant is magically the same formula as before:

$$\det A = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

Cofactor Expansions
Example

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5 \cdot \det \begin{pmatrix} 5 & 1 & 0 \\ -1 & -3 & 2 \\ 4 & 0 & -1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$+ 0 \cdot \det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= 5 \cdot \det \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 0 \cdot \det \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$$

$$= 5 \cdot (-3 - 0) - 1 \cdot (1 - 8)$$

$$= -15 + 7 = -8$$

Recall: the formula

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

is called **cofactor expansion along the first row.** Actually, you can expand cofactors along any row or column you like!

Magical Theorem

The determinant of an $n \times n$ matrix A is

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for any fixed } i$$

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for any fixed } j$$

These formulas are called **cofactor expansion along the ith row**, respectively, **jth column**.

In particular, you get the *same answer* whichever row or column you choose.

Try this with a row or a column with a lot of zeros.

Cofactor Expansion
Example

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

It looks easiest to expand along the third column:

$$\det A = 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} - 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 2 - 1 = 1$$

- ▶ In general, computing a determinant by cofactor expansion is slower than by row reduction.
- ▶ It makes sense to expand by cofactors if you have a row or column with a lot of zeros.
- ▶ Also if your matrix has unknowns in it, since those are hard to row reduce (you don't know where the pivots are).

You can also use more than one method; for example:

- ▶ Use cofactors on a 4×4 matrix but compute the minors using the 3×3 formula.
- ▶ Do row operations to produce a row/column with lots of zeros, then expand cofactors (but keep track of how you changed the determinant!).

Example:

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} \xrightarrow{R_2 = R_2 + 2R_3} \det \begin{pmatrix} 5 & 1 & 0 \\ 7 & 3 & 0 \\ 4 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{\text{3rd column}} (-1) \det \begin{pmatrix} 5 & 1 \\ 7 & 3 \end{pmatrix} = -8$$

Poll

$$\det \begin{pmatrix} 0 & 7 & 2 & 9 & 8 \\ 1 & 3 & 2 & 7 & 4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix} = ?$$

A. -84 **B.** -28 **C.** -7 **D.** 0 **E.** 7 **F.** 28 **G.** 84

Repeatedly expanding along the first row, you get:

$$\det \begin{pmatrix} 0 & 7 & 2 & 9 & 8 \\ 1 & 3 & 2 & 7 & 4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix} = -1 \cdot \det \begin{pmatrix} 7 & 2 & 9 & 8 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 5 \end{pmatrix}$$

$$= (-1) \cdot 7 \cdot \det \begin{pmatrix} 0 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 5 \end{pmatrix} = (-1) \cdot 7 \cdot 3 \cdot \det \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$= (-1) \cdot 7 \cdot 3 \cdot 2 \cdot 2 = -84.$$

A General Formula for the Inverse

For 2×2 matrices we had a nice formula for the inverse:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix}.$$

Theorem

This last formula works for any $n \times n$ invertible matrix A:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix} = \frac{1}{\det A} \left(C_{ij} \right)^{T}$$

(3, 1) entry

Note that the cofactors are "transposed": the (i, j) entry of the matrix is C_{ji}.

The proof uses Cramer's rule.

A Formula for the Inverse
Example

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The minors are:

$$A_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{23} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$A_{31} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_{32} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A_{33} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The cofactors are (don't forget to multiply by $(-1)^{i+j}$):

$$
\begin{array}{lll}
C_{11} = -1 & C_{12} = 1 & C_{13} = -1 \\
C_{21} = 1 & C_{22} = -1 & C_{23} = -1 \\
C_{31} = -1 & C_{32} = -1 & C_{33} = 1
\end{array}
$$

The determinant is (expanding along the first row):

$$\det A = 1 \cdot C_{11} + 0 \cdot C_{12} + 1 \cdot C_{13} = -2$$

Compute A^{-1}, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The inverse is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \checkmark$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix}$$

That was a lot of work! It's way easier to compute inverses by row reduction.

▶ The formula is good for error estimates: the only division is by the determinant, so if your determinant is tiny, your error bars are large.

▶ It's also useful if your matrix has unknowns in it.

▶ It's part of a larger picture in the theory.

Summary

We have several ways to compute the determinant of a matrix.

- Special formulas for 2×2 and 3×3 matrices.

  These work great for small matrices.

- Cofactor expansion.

  This is perfect when there is a row or column with a lot of zeros, or if your matrix has unknowns in it.

- Row reduction.

  This is the way to go when you have a big matrix which doesn't have a row or column with a lot of zeros.

- Any combination of the above.

  Cofactor expansion is recursive, but you don't have to use cofactor expansion to compute the determinants of the minors! Or you can do row operations and then a cofactor expansion.

Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvalues and Eigenvectors

A Biology Question

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have $0, 6, 8$ baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

$$f_n = \text{first-year rabbits in year } n$$
$$s_n = \text{second-year rabbits in year } n$$
$$t_n = \text{third-year rabbits in year } n$$

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $\boxed{Av_n = v_{n+1}.}$ ⟵ **difference equation**

If you know v_0, what is v_{10}?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$

This makes it easy to compute examples by computer: [interactive]

v_0	v_{10}	v_{11}
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.

2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$

Translation: 2 is an eigenvalue, and $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector!

Eigenvectors and Eigenvalues

Definition

Let A be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of A is a *nonzero* vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$.   In other words, Av is a multiple of v.

2. An **eigenvalue** of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a *nontrivial* solution.

If $Av = \lambda v$ for $v \neq 0$, we say λ is the **eigenvalue for** v, and v is an **eigenvector for** λ.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

Hence v is an eigenvector of A, with eigenvalue $\lambda = 2$.

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence v is an eigenvector of A, with eigenvalue $\lambda = 4$.

Poll

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector with eigenvalue 2

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ eigenvector with eigenvalue 0

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvector with eigenvalue 0

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ not an eigenvector

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

In other words, does $Av = 3v$ have a nontrivial solution?

         ...does $Av - 3v = 0$ have a nontrivial solution?

         ...does $(A - 3I)v = 0$ have a nontrivial solution?

We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric form: $x = -4y$; parametric vector form: $\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Does there exist an eigenvector with eigenvalue $\lambda = 3$? Yes! Any nonzero multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Check:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} -4 \\ 1 \end{pmatrix}. \quad \checkmark$$

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The λ-**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ, plus the zero vector:

$$\begin{aligned} \lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \mathrm{Nul}(A - \lambda I). \end{aligned}$$

Since the λ-eigenspace is a null space, it is a *subspace* of $\mathbf{R}^n$.

How do you find a basis for the λ-eigenspace? Parametric vector form!

Eigenspaces

Example

Find a basis for the 3-eigenspace of

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}.$$

We have to solve the matrix equation $A - 3I_2 = 0$.

$$A - 3I_2 = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric form}} x = -4y$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\}.$$

Find a basis for the 2-eigenspace of

λ

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$$A - 2I = \begin{pmatrix} \frac{3}{2} & 0 & 3 \\ -\frac{3}{2} & 0 & -3 \\ -\frac{3}{2} & 0 & -3 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric form}} x = -2z$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Find a basis for the $\frac{1}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$$A - \frac{1}{2}I = \begin{pmatrix} 3 & 0 & 3 \\ -\frac{3}{2} & \frac{3}{2} & -3 \\ -\frac{3}{2} & 0 & -\frac{3}{2} \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x = -z \\ y = z \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$
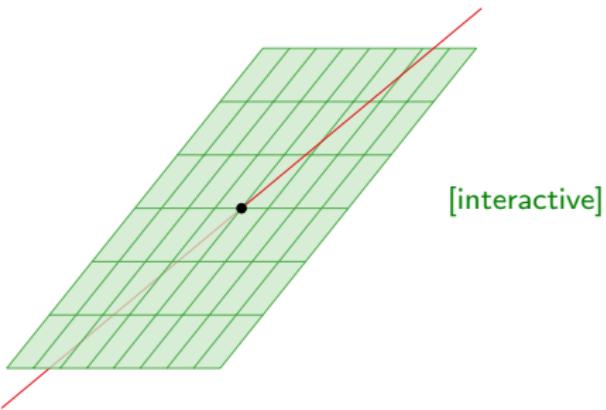
$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

We computed bases for the 2-eigenspace and the 1/2-eigenspace:

2-eigenspace: $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\qquad \frac{1}{2}$-eigenspace: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Hence the 2-eigenspace is a plane and the 1/2-eigenspace is a line.

[interactive]

Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\text{Nul}(A - \lambda I) \neq \{0\}$.

2. In this case, finding a basis for the λ-eigenspace of A means finding a basis for $\text{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.

3. The eigenvectors with eigenvalue λ are the nonzero elements of $\text{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

We've seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix *is not a row reduction problem!* We'll see how to do it in general next time. For now:

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Why? $\text{Nul}(A - \lambda I) \neq \{0\}$ if and only if $A - \lambda I$ is not invertible, if and only if $\det(A - \lambda I) = 0$.

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \lambda I_4 = \begin{pmatrix} 3-\lambda & 4 & 1 & 2 \\ 0 & -1-\lambda & -2 & 7 \\ 0 & 0 & 8-\lambda & 12 \\ 0 & 0 & 0 & -3-\lambda \end{pmatrix}.$$

The determinant is $(3 - \lambda)(-1 - \lambda)(8 - \lambda)(-3 - \lambda)$, which is zero exactly when $\lambda = 3, -1, 8,$ or -3.

Fact: A is invertible if and only if 0 is not an eigenvalue of A.

Why?

$$0 \text{ is an eigenvalue of } A \iff Ax = 0x \text{ has a nontrivial solution}$$
$$\iff Ax = 0 \text{ has a nontrivial solution}$$
$$\iff A \text{ is not invertible.}$$

invertible matrix theorem

Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Why? If $k = 2$, this says v_2 can't lie on the line through v_1.

But the line through v_1 is contained in the λ_1-eigenspace, and v_2 does not have eigenvalue λ_1.

In general: see §5.1 (or work it out for yourself; it's not too hard).

Consequence: An $n \times n$ matrix has at most n distinct eigenvalues.

We have a couple of new ways of saying "A is invertible" now:

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T \colon \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.

2. T is invertible.
3. The reduced row echelon form of A is I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial one.
6. $\text{Nul}(A) = \{0\}$.
7. $\text{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.

11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\text{Col}\,A = \mathbf{R}^m$.
15. $\dim \text{Col}\,A = m$.
16. $\text{rank}\,A = m$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

20. The determinant of A is *not* equal to zero.
21. The number 0 is *not* an eigenvalue of A.

Summary

- **Eigenvectors** and **eigenvalues** are the most important concepts in this course.

- Eigenvectors are by definition nonzero; eigenvalues may be zero.

- The eigenvalues of a triangular matrix are the diagonal entries.

- A matrix is invertible if and only if zero is not an eigenvalue.

- Eigenvectors with distinct eigenvalues are linearly independent.

- The λ-eigenspace is the set of all λ-eigenvectors, plus the zero vector.

- You can compute a basis for the λ-eigenspace by finding the parametric vector form of the solutions of $(A - \lambda I_n)x = 0$.

Definition

Let A be an $n \times n$ matrix.

1. An **eigenvector** of A is a nonzero vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$.

2. An **eigenvalue** of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

3. If λ is an eigenvalue of A, the λ-**eigenspace** is the solution set of $(A - \lambda I_n)x = 0$.
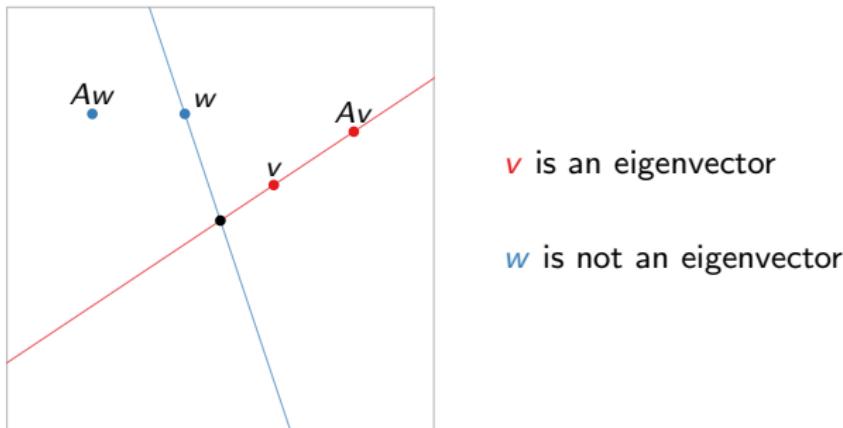
Eigenvectors, geometrically

An eigenvector of a matrix A is a nonzero vector v such that:

- Av is a multiple of v, which means
- Av is collinear with v, which means
- Av and v are *on the same line through the origin*.


v is an eigenvector

w is not an eigenvector

Let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

v is an eigenvector with eigenvalue -1.

[interactive]

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

w is an eigenvector with eigenvalue 1.

[interactive]

Let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

u is *not* an eigenvector.

[interactive]

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

Neither is z.

[interactive]

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

The 1-eigenspace is L
   (all the vectors x where $Ax = x$).
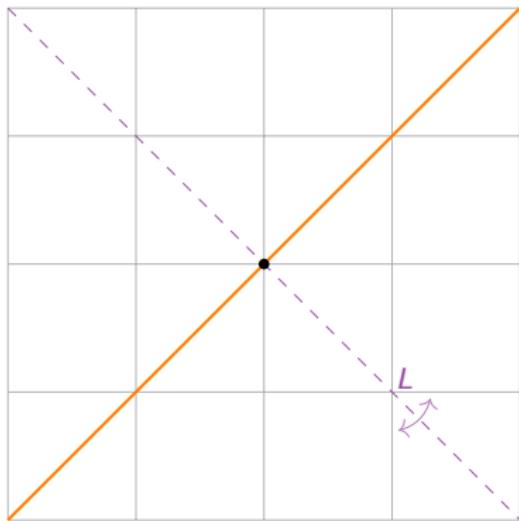
[interactive]

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be reflection over the line L defined by $y = -x$, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

The (-1)-eigenspace is the line $y = x$ (all the vectors x where $Ax = -x$).
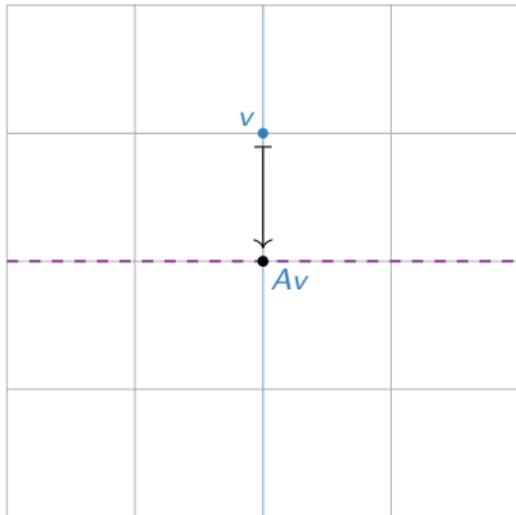
[interactive]

Let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

v is an eigenvector with eigenvalue 0.

[interactive]

Let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

w is an eigenvector with eigenvalue 1.

[interactive]

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

u is *not* an eigenvector.

[interactive]

Let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

Neither is z.

[interactive]

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

The 1-eigenspace is the x-axis
  (all the vectors x where $Ax = x$).

[interactive]

Let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vertical projection onto the x-axis, and let A be the matrix for T.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

The 0-eigenspace is the y-axis
   (all the vectors x where $Ax = 0x$).

[interactive]
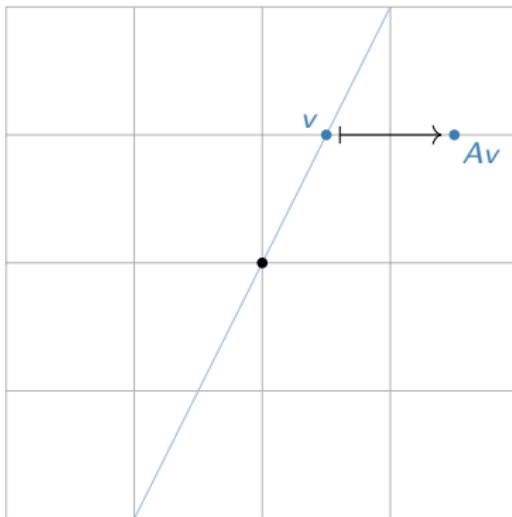
Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

Vectors v above the x-axis are moved right but not up...
 so they're not eigenvectors.
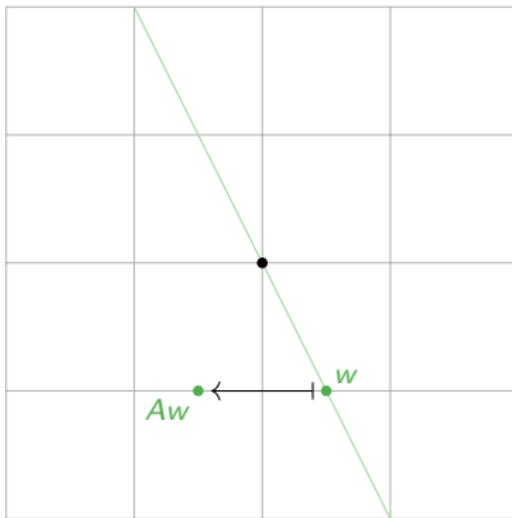
[interactive]

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

Vectors w below the x-axis are moved left but not down. . .
  so they're not eigenvectors
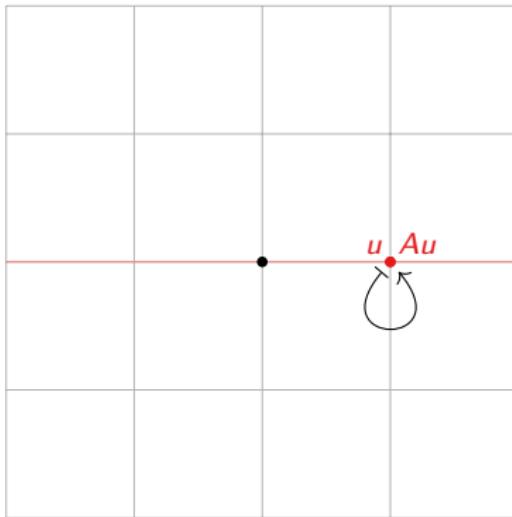
[interactive]

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

u is an eigenvector with eigenvalue 1.
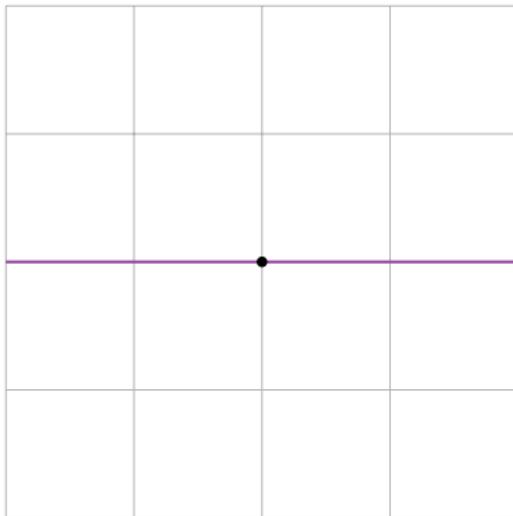
[interactive]

Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!

Does anyone see any eigenvectors (vectors that don't move off their line)?

The 1-eigenspace is the x-axis (all the vectors x where $Ax = x$).

[interactive]

Let
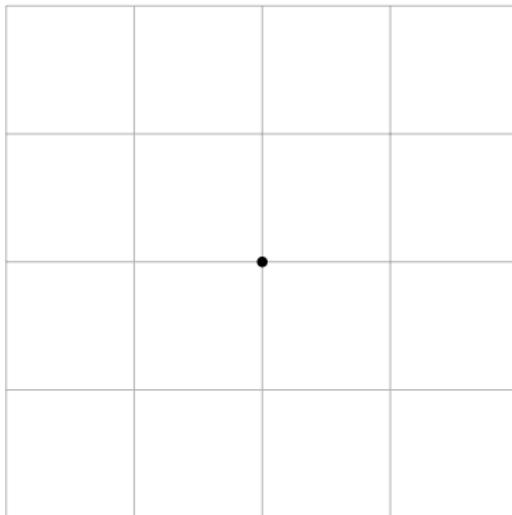
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

so $T(x) = Ax$ is a shear in the x-direction.

Question: What are the eigenvalues and eigenspaces of A? No computations!
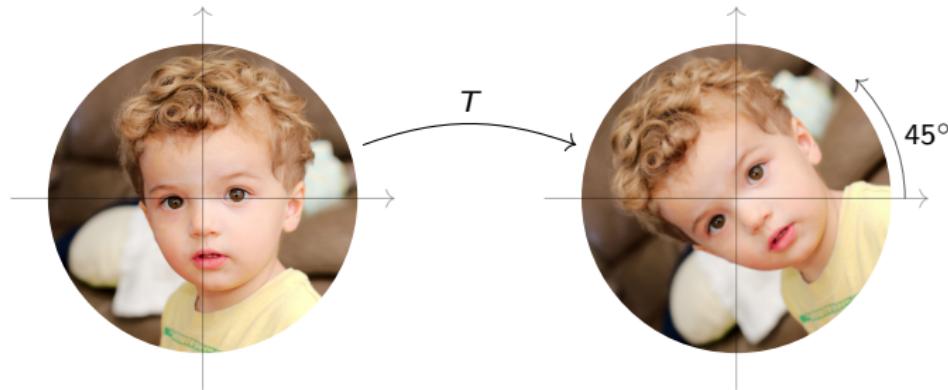
Does anyone see any eigenvectors (vectors that don't move off their line)?

There are no other eigenvectors.

[interactive]

Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be counterclockwise rotation by $45°$, and let A be the matrix for T.

Poll

Find an eigenvector of A without doing any computations.

A. Okay.   **B.** No way.

Answer: **B.** No way. There are no eigenvectors!   [interactive]

Section 5.2

The Characteristic Polynomial

The Characteristic Polynomial

Let A be a square matrix.

λ is an eigenvalue of A $\iff$ $Ax = \lambda x$ has a nontrivial solution

$\iff$ $(A - \lambda I)x = 0$ has a nontrivial solution

$\iff$ $A - \lambda I$ is not invertible

$\iff$ $\det(A - \lambda I) = 0$.

This gives us a way to compute the eigenvalues of A.

Definition

Let A be a square matrix. The **characteristic polynomial** of A is

$$f(\lambda) = \det(A - \lambda I).$$

The **characteristic equation** of A is the equation

$$f(\lambda) = \det(A - \lambda I) = 0.$$

> **Important**
>
> The eigenvalues of A are the roots of the characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.

Question: What are the eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$f(\lambda) = \det(A - \lambda I) = \det\left[\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \det\begin{pmatrix} 5 - \lambda & 2 \\ 2 & 1 - \lambda \end{pmatrix}$$

$$= (5 - \lambda)(1 - \lambda) - 2 \cdot 2$$

$$= \lambda^2 - 6\lambda + 1.$$

The eigenvalues are the roots of the characteristic polynomial, which we can find using the quadratic formula:

$$\lambda = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}.$$

Question: What is the characteristic polynomial of
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}?$$

Answer:
$$f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc$$

$$= \lambda^2 - (a + d)\lambda + (ad - bc)$$

What do you notice about $f(\lambda)$?

- The constant term is $\det(A)$, which is zero if and only if $\lambda = 0$ is a root.
- The linear term $-(a + d)$ is the negative of the sum of the diagonal entries of A.

Definition
The **trace** of a square matrix A is $\text{Tr}(A) = $ sum of the diagonal entries of A.

Shortcut
The characteristic polynomial of a 2×2 matrix A is
$$f(\lambda) = \lambda^2 - \text{Tr}(A)\,\lambda + \det(A).$$

Question: What are the eigenvalues of the rabbit population matrix

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}?$$

Answer: First we find the characteristic polynomial:

$$f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 6 & 8 \\ \frac{1}{2} & -\lambda & 0 \\ 0 & \frac{1}{2} & -\lambda \end{pmatrix}$$

$$= 8\left(\frac{1}{4} - 0 \cdot -\lambda\right) - \lambda\left(\lambda^2 - 6 \cdot \frac{1}{2}\right)$$

$$= -\lambda^3 + 3\lambda + 2.$$

We know from before that one eigenvalue is $\lambda = 2$: indeed, $f(2) = -8 + 6 + 2 = 0$. Doing polynomial long division, we get:

$$\frac{-\lambda^3 + 3\lambda + 2}{\lambda - 2} = -\lambda^2 - 2\lambda - 1 = -(\lambda + 1)^2.$$

Hence $\lambda = -1$ is also an eigenvalue.

Factoring the Characteristic Polynomial

It's easy to factor quadraic polynomials:

$$x^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

It's less easy to factor cubics, quartics, and so on:

$$x^3 + bx^2 + cx + d = 0 \implies x = \text{???}$$
$$x^4 + bx^3 + cx^2 + dx + e = 0 \implies x = \text{???}$$

Read about factoring polynomials by hand in §5.2.

Summary

We did two different things today.

First we talked about the geometry of eigenvalues and eigenvectors:

- Eigenvectors are vectors v such that v and Av are on the same line through the origin.
- You can pick out the eigenvectors geometrically if you have a picture of the associated transformation.

Then we talked about characteristic polynomials:

- We learned to find the eigenvalues of a matrix by computing the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.
- For a 2×2 matrix A, the characteristic polynomial is just

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

Section 5.4

Diagonalization

Many real-word linear algebra problems have the form:

$$v_1 = Av_0, \quad v_2 = Av_1 = A^2 v_0, \quad v_3 = Av_2 = A^3 v_0, \quad \ldots \quad v_n = Av_{n-1} = A^n v_0.$$

This is called a **difference equation**.

Our toy example about rabbit populations had this form.

The question is, what happens to v_n as $n \to \infty$?

- Taking powers of diagonal matrices is easy!
- Taking powers of *diagonalizable* matrices is still easy!
- Diagonalizing a matrix is an eigenvalue problem.

Powers of Diagonal Matrices

If D is diagonal, then D^n is also diagonal; its diagonal entries are the nth powers of the diagonal entries of D:

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^3 = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}, \quad \cdots \quad D^n = \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix}.$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad D^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{27} \end{pmatrix},$$

$$\cdots \quad D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{pmatrix}$$

What if A is not diagonal?

Example

Let $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$. Compute A^n, using

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

We compute:

$$A^2 = (CDC^{-1})(CDC^{-1}) = CD(C^{-1}C)DC^{-1} = CDIDC^{-1} = CD^2C^{-1}$$
$$A^3 = (CDC^{-1})(CD^2C^{-1}) = CD(C^{-1}C)D^2C^{-1} = CDID^2C^{-1} = CD^3C^{-1}$$

$$\vdots$$

$$A^n = CD^nC^{-1}$$

Closed formula in terms of n:
easy to compute

Therefore

$$A^n = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} \frac{1}{-2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2^n + (-1)^n & 2^n + (-1)^{n+1} \\ 2^n + (-1)^{n+1} & 2^n + (-1)^n \end{pmatrix}.$$

Similar Matrices

Definition
Two $n \times n$ matrices are **similar** if there exists an invertible $n \times n$ matrix C such that $A = CBC^{-1}$.

Fact: if two matrices are similar then so are their powers:

$$A = CBC^{-1} \quad \implies \quad A^n = CB^nC^{-1}.$$

Fact: if A is similar to B and B is similar to D, then A is similar to D.

$$A = CBC^{-1}, \quad B = EDE^{-1} \quad \implies \quad A = C(EDE^{-1})C^{-1} = (CE)D(CE)^{-1}.$$

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = CDC^{-1} \quad \text{for } D \text{ diagonal.}$$

Important

If $A = CDC^{-1}$ for $D = \begin{pmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{pmatrix}$ then

$$A^k = CD^K C^{-1} = C \begin{pmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{pmatrix} C^{-1}.$$

So diagonalizable matrices are easy to raise to any power.

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In this case, $A = CDC^{-1}$ for

$$
C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad
D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},
$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary ⟵——— a theorem that follows easily from another theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

The Corollary is true because eigenvectors with distinct eigenvalues are always linearly independent. We will see later that a diagonalizable matrix need not have n distinct eigenvalues though.

Diagonalization

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In this case, $A = CDC^{-1}$ for

$$
C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}
\qquad
D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},
$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Note that the decomposition is not unique: you can reorder the eigenvalues and eigenvectors.

$$
A = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^{-1} = \begin{pmatrix} | & | \\ v_2 & v_1 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} | & | \\ v_2 & v_1 \\ | & | \end{pmatrix}^{-1}
$$

Question: What does the Diagonalization Theorem say about the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}?$$

This is a triangular matrix, so the eigenvalues are the diagonal entries $1, 2, 3$.

A diagonal matrix just scales the coordinates by the diagonal entries, so we can take our eigenvectors to be the unit coordinate vectors e_1, e_2, e_3. Hence the Diagonalization Theorem says

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It doesn't give us anything new because the matrix was already diagonal!

> A diagonal matrix D is diagonalizable! It is similar to itself:
>
> $$D = I_n D I_n^{-1}.$$

Diagonalization
Example

Problem: Diagonalize $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$.

The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \operatorname{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2).$$

Therefore the eigenvalues are -1 and 2. Let's compute some eigenvectors:

$$(A + 1I)x = 0 \iff \begin{pmatrix} 3/2 & 3/2 \\ 3/2 & 3/2 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = -y$, so $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue -1.

$$(A - 2I)x = 0 \iff \begin{pmatrix} -3/2 & 3/2 \\ 3/2 & -3/2 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = y$, so $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 2.

The eigenvectors v_1, v_2 are linearly independent, so the Diagonalization Theorem says

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

The characteristic polynomial is
$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2).$$

Therefore the eigenvalues are 1 and 2, with respective multiplicities 2 and 1.
Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 3 & -3 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric vector form is

$$\begin{matrix} x = y \\ y = y \\ z = z \end{matrix} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Hence a basis for the 1-eigenspace is

$$\mathcal{B}_1 = \{v_1, v_2\} \quad \text{where} \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Problem: Diagonalize $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

Now let's compute the 2-eigenspace:

$$(A - 2I)x = 0 \iff \begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 1 & -1 & -1 \end{pmatrix} x = 0 \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} x = 0$$

The parametric form is $x = 3z, y = 2z$, so an eigenvector with eigenvalue 2 is

$$v_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The eigenvectors v_1, v_2, v_3 are linearly independent: v_1, v_2 form a basis for the 1-eigenspace, and v_3 is not contained in the 1-eigenspace. Therefore the Diagonalization Theorem says

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note: In this case, there are three linearly independent eigenvectors, but only two distinct eigenvalues.

Problem: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

This is an upper-triangular matrix, so the only eigenvalue is 1. Let's compute the 1-eigenspace:

$$(A - I)x = 0 \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = 0.$$

This is row reduced, but has only one free variable x; a basis for the 1-eigenspace is $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$. So *all eigenvectors* of A are multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Conclusion: A has only one linearly independent eigenvector, so by the "only if" part of the diagonalization theorem, A is not diagonalizable.

Poll

Which of the following matrices are diagonalizable, and why?

A. $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ C. $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ D. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Matrix A is not diagonalizable: its only eigenvalue is 1, and its 1-eigenspace is spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Similarly, matrix C is not diagonalizable.

Matrix B is diagonalizable because it is a 2×2 matrix with distinct eigenvalues.

Matrix D is already diagonal!

How to diagonalize a matrix A:

1. Find the eigenvalues of A using the characteristic polynomial.

2. For each eigenvalue λ of A, compute a basis $\mathcal{B}_\lambda$ for the λ-eigenspace.

3. If there are fewer than n total vectors in the union of all of the eigenspace bases $\mathcal{B}_\lambda$, then the matrix is not diagonalizable.

4. Otherwise, the n vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = CDC^{-1}$ for

$$
C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},
$$

where λ_i is the eigenvalue for v_i.

Why is the Diagonalization Theorem true?

A diagonalizable implies A has n linearly independent eigenvectors: Suppose $A = CDC^{-1}$, where D is diagonal with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ be the columns of C. They are linearly independent because C is invertible. So $Ce_i = v_i$, hence $C^{-1}v_i = e_i$.

$$Av_i = CDC^{-1}v_i = CDe_i = C(\lambda_i e_i) = \lambda_i Ce_i = \lambda_i v_i.$$

Hence v_i is an eigenvector of A with eigenvalue λ_i. So the columns of C form n linearly independent eigenvectors of A, and diagonal entries of D are the eigenvalues.

A has n linearly independent eigenvectors implies A is diagonalizable: Suppose A has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let C be the invertible matrix with columns $v_1, v_2, \ldots, v_n$. Let $D = C^{-1}AC$.

$$De_i = C^{-1}ACe_i = C^{-1}Av_i = C^{-1}(\lambda_i v_i) = \lambda_i C^{-1}v_i = \lambda_i e_i.$$

Hence D is diagonal, with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$. Solving $D = C^{-1}AC$ for A gives $A = CDC^{-1}$.

Algebraic Multiplicity

Definition
The **(algebraic) multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

This is not a very interesting notion *yet*. It will become interesting when we also define *geometric* multiplicity later.

Example
In the rabbit population matrix, $f(\lambda) = -(\lambda - 2)(\lambda + 1)^2$, so the algebraic multiplicity of the eigenvalue 2 is 1, and the algebraic multiplicity of the eigenvalue -1 is 2.

Example
In the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, $f(\lambda) = (\lambda - (3 - 2\sqrt{2}))(\lambda - (3 + 2\sqrt{2}))$, so the algebraic multiplicity of $3 + 2\sqrt{2}$ is 1, and the algebraic multiplicity of $3 - 2\sqrt{2}$ is 1.

Non-Distinct Eigenvalues

Definition
Let λ be an eigenvalue of a square matrix A. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem
Let λ be an eigenvalue of a square matrix A. Then

$1 \leq$ (the geometric multiplicity of λ) $\leq$ (the algebraic multiplicity of λ).

The proof is beyond the scope of this course.

Corollary
Let λ be an eigenvalue of a square matrix A. If the algebraic multiplicity of λ is 1, then the geometric multiplicity is also 1: the eigenspace is a *line*.

The Diagonalization Theorem (Alternate Form)
Let A be an $n \times n$ matrix. The following are equivalent:

1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and for each eigenvalue, *the geometric multiplicity equals the algebraic multiplicity*.

Example

If A has n distinct eigenvalues, then the algebraic multiplicity of each equals 1, hence so does the geometric multiplicity, and therefore A is diagonalizable.

For example, $A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$ has eigenvalues -1 and 2, so it is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -(\lambda - 1)^2(\lambda - 2).$$

The algebraic multiplicities of 1 and 2 are 2 and 1, respectively. They sum to 3.

We showed before that the geometric multiplicity of 1 is 2 (the 1-eigenspace has dimension 2). The eigenvalue 2 automatically has geometric multiplicity 1.

Hence the geometric multiplicities add up to 3, so A is diagonalizable.

Example

The matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $f(\lambda) = (\lambda - 1)^2$.

It has one eigenvalue 1 of algebraic multiplicity 2.

We showed before that the geometric multiplicity of 1 is 1 (the 1-eigenspace has dimension 1).

Since the geometric multiplicity is smaller than the algebraic multiplicity, the matrix is *not* diagonalizable.

Summary

- A matrix A is **diagonalizable** if it is similar to a diagonal matrix D: $A = CDC^{-1}$.
- It is easy to take powers of diagonalizable matrices: $A^r = CD^r C^{-1}$.
- An $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$, in which case $A = CDC^{-1}$ for

$$
C = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}
\qquad
D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.
$$

- If A has n distinct eigenvalues, then it is diagonalizable.
- The **geometric multiplicity** of an eigenvalue λ is the dimension of the λ-eigenspace.
- $1 \leq$ (geometric multiplicity) $\leq$ (algebraic multiplicity).
- An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities is n.

Recall: an $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = CDC^{-1} \quad \text{for} \quad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

It is easy to take powers of diagonalizable matrices:

$$A^i = CD^iC^{-1} = C \begin{pmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^i \end{pmatrix} C^{-1}.$$
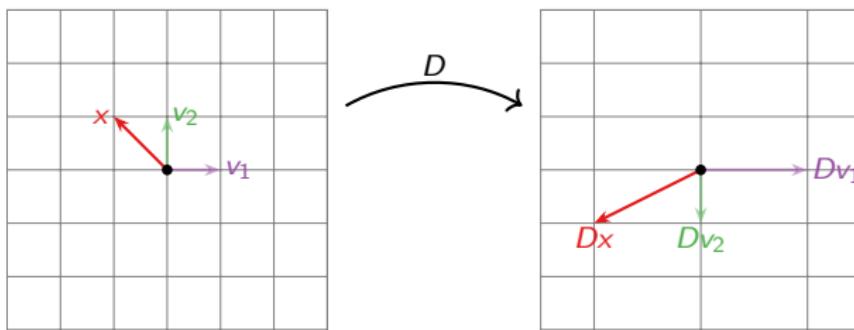
We begin today by discussing the *geometry* of diagonalizable matrices.

A diagonal matrix $D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ just scales the coordinate axes:

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This is easy to visualize:

$$x = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \Longrightarrow \qquad Dx = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

Geometry of Diagonal*izable* Matrices

We had this example last time: $A = CDC^{-1}$ for

$$A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$
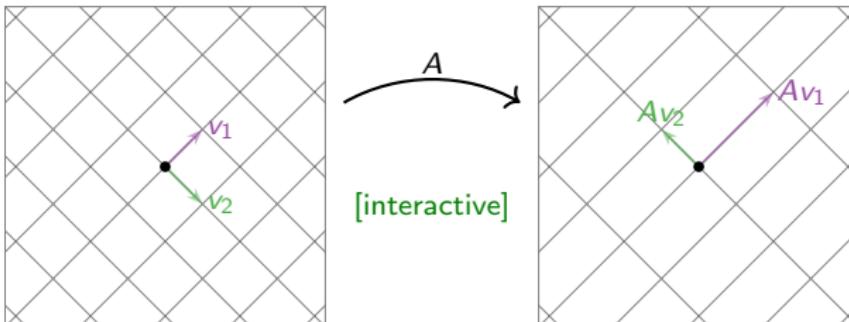
The eigenvectors of A are $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalues 2 and -1.

The eigenvectors form a *basis* for $\mathbf{R}^2$ because they're linearly independent.

Any vector can be written as a linear combination of basis vectors:

$$x = a_1 v_1 + a_2 v_2 \implies Ax = A(a_1 v_1 + a_2 v_2) = a_1 A v_1 + a_2 A v_2 = 2a_1 v_1 - a_2 v_2.$$

Conclusion: A scales the "v_1-direction" by 2 and the "v_2-direction" by -1.
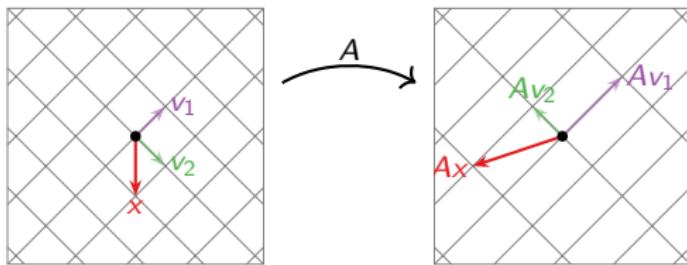
Geometry of Diagonal*izable* Matrices
Continued
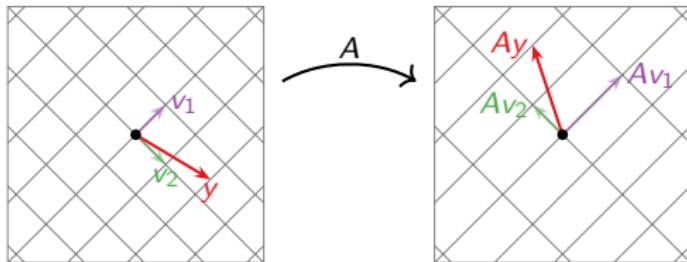
Example: $x = \begin{pmatrix} 0 \\ -2 \end{pmatrix} = -1v_1 + 1v_2$

$$Ax = -1Av_1 + 1Av_2 = -2v_1 + -1v_2 = -2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}.$$

Example: $y = \frac{1}{2}\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \frac{1}{2}v_1 + 2v_2$

$$Ay = \frac{1}{2}Av_1 + 2Av_2 = 1v_1 + -2v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

We motivated diagonalization by taking powers:

$$A^i = CD^iC^{-1} = C \begin{pmatrix} \lambda_1^i & 0 & \cdots & 0 \\ 0 & \lambda_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^i \end{pmatrix} C^{-1}.$$

This lets us compute powers of matrices easily. How to visualize this?

$$A^n v = A(A(A \cdots (Av)) \cdots)$$

Multiplying a vector v by A^n means repeatedly multiplying by A.

Dynamics of Diagonalizable Matrices

Example

$$A = \frac{1}{10}\begin{pmatrix} 11 & 6 \\ 9 & 14 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 2/3 & 1 \\ -1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Eigenvectors of A are $v_1 = \begin{pmatrix} 2/3 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalues 2 and 1/2.

$$A(a_1 v_1 + a_2 v_2) = 2a_1 v_1 + \frac{1}{2} a_2 v_2$$

$$A^2(a_1 v_1 + a_2 v_2) = 4a_1 v_1 + \frac{1}{4} a_2 v_2$$

$$A^3(a_1 v_1 + a_2 v_2) = 8a_1 v_1 + \frac{1}{8} a_2 v_2$$

$$\vdots$$

$$A^n(a_1 v_1 + a_2 v_2) = 2^n a_1 v_1 + \frac{1}{2^n} a_2 v_2$$

What does repeated application of A do geometrically?

It makes the "v_1-coordinate" very big, and the "v_2-coordinate" very small.

[interactive]

Dynamics of Diagonalizable Matrices

Another Example

$$A = \frac{1}{6}\begin{pmatrix} 5 & -1 \\ -2 & 4 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} -1 & 1/2 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Eigenvectors of A are $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ with eigenvalues 1 and $1/2$.

$$A(a_1 v_1 + a_2 v_2) = a_1 v_1 + \frac{1}{2} a_2 v_2$$

$$A^2(a_1 v_1 + a_2 v_2) = a_1 v_1 + \frac{1}{4} a_2 v_2$$

$$A^3(a_1 v_1 + a_2 v_2) = a_1 v_1 + \frac{1}{8} a_2 v_2$$

$$\vdots$$

$$A^n(a_1 v_1 + a_2 v_2) = a_1 v_1 + \frac{1}{2^n} a_2 v_2$$

What does repeated application of A do geometrically?

It "sucks everything into the 1-eigenspace."

[interactive]

Dynamics of Diagonalizable Matrices

$$A = \frac{1}{30} \begin{pmatrix} 12 & 2 \\ 3 & 13 \end{pmatrix} = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 2/3 & -1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}.$$

Poll

What does repeated application of A do geometrically?

  A. Sucks all vectors into a line.

  B. Sucks all vectors into the origin.

  C. Shoots all vectors away from a line.

  D. Shoots all vectors away from the origin.

B. Since both eigenvalues are less than 1, the matrix A scales *both* directions towards the origin.

[interactive]

Section 5.5

Complex Eigenvalues
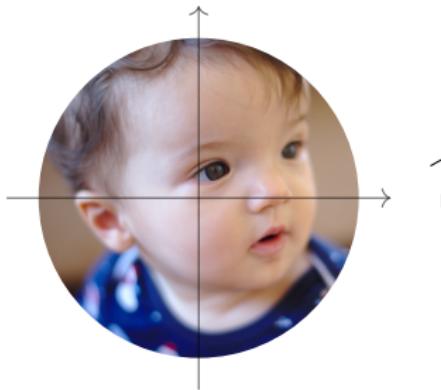
A Matrix with No Eigenvectors

Consider the matrix for the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:
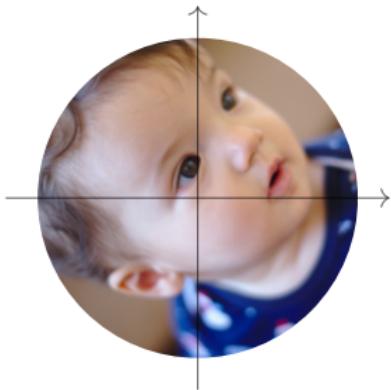
$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has no eigenvectors, as you can see geometrically: [interactive]

A

no nonzero vector x
is collinear with Ax

or algebraically:

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \sqrt{2}\,\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

Complex Numbers

It makes us sad that -1 has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

Mathematician's solution: we're just not using enough numbers! We're going to declare by *fiat* that there exists a square root of -1.

Definition
The number i is defined such that $i^2 = -1$.
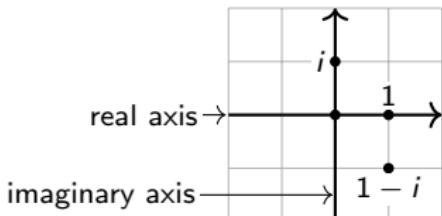
Once we have i, we have to allow numbers like $a + bi$ for real numbers a, b.

Definition
A *complex number* is a number of the form $a + bi$ for a, b in $\mathbf{R}$. The set of all complex numbers is denoted $\mathbf{C}$.

Note $\mathbf{R}$ is contained in $\mathbf{C}$: they're the numbers $a + 0i$.

We can identify $\mathbf{C}$ with $\mathbf{R}^2$ by $a + bi \longleftrightarrow \binom{a}{b}$. So when we draw a picture of $\mathbf{C}$, we draw the plane:

Operations on Complex Numbers

Addition: $(2 - 3i) + (-1 + i) = 1 - 2i$.

Multiplication: $(2 - 3i)(-1 + i) = 2(-1) + 2i + 3i - 3i^2 = -2 + 5i + 3 = 1 + 5i$.

Complex conjugation: $\overline{a + bi} = a - bi$ is the **complex conjugate** of $a + bi$.
Check: $\overline{z + w} = \bar{z} + \bar{w}$ and $\overline{zw} = \bar{z} \cdot \bar{w}$.

Absolute value: $|a + bi| = \sqrt{a^2 + b^2}$. This is a *real* number.
Note: $(a + bi)(\overline{a + bi}) = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$. So $|z| = \sqrt{z\bar{z}}$.
Check: $|zw| = |z| \cdot |w|$.

Division by a nonzero real number: $\dfrac{a + bi}{c} = \dfrac{a}{c} + \dfrac{b}{c}i$.

Division by a nonzero complex number: $\dfrac{z}{w} = \dfrac{z\bar{w}}{w\bar{w}} = \dfrac{z\bar{w}}{|w|^2}$.

Example:
$$\frac{1 + i}{1 - i} = \frac{(1 + i)^2}{1^2 + (-1)^2} = \frac{1 + 2i + i^2}{2} = i.$$

Real and imaginary part: $\mathrm{Re}(a + bi) = a \qquad \mathrm{Im}(a + bi) = b$.

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a complex root of f, then so is $\overline{\lambda}$:

$$0 = \overline{f(\lambda)} = \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0}$$
$$= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f\left(\overline{\lambda}\right).$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

For instance, if $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$ then
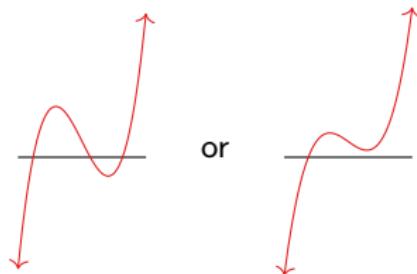
$$\lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2}(1 \pm i) = \frac{1 \pm i}{\sqrt{2}}.$$

Note the roots are complex conjugates if b, c are real.

Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:

or

respectively.

A Matrix *with* an Eigenvector

Every matrix is guaranteed to have *complex* eigenvalues and eigenvectors. Using rotation by $\pi/4$ from before:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda = \frac{1 \pm i}{\sqrt{2}}.$$

Let's compute an eigenvector for $\lambda = (1+i)/\sqrt{2}$:

$$A - \lambda I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

The second row is i times the first, so we row reduce:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{divide by } -i/\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, so an eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$. So is *any* nonzero complex scalar multiple of $\begin{pmatrix} i \\ 1 \end{pmatrix}$, for example $(-i/\sqrt{2}) \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$.

A similar computation shows that an eigenvector for $\lambda = (1-i)/\sqrt{2}$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

So is $i \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (you can scale by *complex* numbers).

Conjugate Eigenvectors

For $A = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

the eigenvalue $\dfrac{1+i}{\sqrt{2}}$ has eigenvector $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

the eigenvalue $\dfrac{1-i}{\sqrt{2}}$ has eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Do you notice a pattern?

Fact

Let A be a real square matrix. If λ is a complex eigenvalue with eigenvector v, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$.

Why?

$$Av = \lambda \implies A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}.$$

> Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

Suppose A is a 2×2 matrix and λ is any eigenvalue of A. Then

$$A - \lambda I_2 = \begin{pmatrix} z & w \\ (*) & (*) \end{pmatrix} \implies \begin{pmatrix} -w \\ z \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue λ.

In the previous example, $\lambda = \dfrac{1+i}{\sqrt{2}}$ was an eigenvalue of $A = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and

$$A - \lambda I = \begin{pmatrix} -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} z & w \\ (*) & (*) \end{pmatrix}.$$

So an eigenvector of A corresponding to λ is

$$v = \begin{pmatrix} -w \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}.$$

This was much faster than doing the full 2×2 row reduction in the previous example, and it agrees with our answer.

A 3 × 3 Example

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

$$f(\lambda) = \det \begin{pmatrix} \frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \lambda & 0 \\ 0 & 0 & 2 - \lambda \end{pmatrix} = \boxed{(2 - \lambda)} \left(\lambda^2 - \frac{8}{5}\lambda + 1 \right).$$

This factors out automatically if you expand cofactors along the third row or column

We computed the roots of this polynomial (times 5) before:

$$\lambda = \quad 2, \quad \frac{4 + 3i}{5}, \quad \frac{4 - 3i}{5}.$$

We eyeball an eigenvector with eigenvalue 2 as $(0, 0, 1)$.

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the other eigenvectors, we row reduce:

$$A - \frac{4 + 3i}{5}I = \begin{pmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 0 & 0 & 2 - \frac{4+3i}{5} \end{pmatrix} \xrightarrow{\text{scale rows}} \begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second row is i times the first:

$$\xrightarrow{\text{row replacement}} \begin{pmatrix} -i & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{divide by } -i, \text{swap}} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, $z = 0$, so an eigenvector is $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$. Therefore, an

eigenvector with conjugate eigenvalue $\dfrac{4 - 3i}{5}$ is $\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Summary

- Diagonal matrices are easy to understand geometrically.
- Diagonalizable matrices behave like diagonal matrices, except with respect to a basis of eigenvectors.
- Repeatedly multiplying by a matrix leads to fun pictures.
- One can do arithmetic with complex numbers just like real numbers: add, subtract, multiply, divide.
- An $n \times n$ matrix always exactly has complex n eigenvalues, counted with (algebraic) multiplicity.
- The complex eigenvalues and eigenvectors of a *real* matrix come in complex conjugate pairs:

$$Av = \lambda v \quad \implies \quad A\bar{v} = \bar{\lambda}\bar{v}.$$

Section 5.6

Stochastic Matrices and PageRank

Definition

A square matrix A is **stochastic** if all of its entries are nonnegative, and the sum of the entries of each column is 1.

We say A is **positive** if all of its entries are positive.

These arise very commonly in modeling of probabalistic phenomena (Markov chains).

> You'll be responsible for knowing basic facts about stochastic matrices, the Perron–Frobenius theorem, and PageRank, but we will not cover them in depth.

Red Box has kiosks all over where you can rent movies. You can return them to any other kiosk. Let A be the matrix whose ij entry is the probability that a customer renting a movie from location j returns it to location i. For example, if there are three locations, maybe

$$A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}.$$

30% probability a movie rented from location 3 gets returned to location 2

The columns sum to 1 because there is a 100% chance that the movie will get returned to *some* location. This is a positive stochastic matrix.

Note that, if $v = (x, y, z)$ represents the number of movies at the three locations, then (assuming the number of movies is large), Red Box will have approximately

$$Av = A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} .3x + .4y + .5z \\ .3x + .4y + .3z \\ .4x + .2y + .2z \end{pmatrix}$$

"The number of movies returned to location 2 will be (on average):
30% of the movies from location 1;
40% of the movies from location 2;
30% of the movies from location 3"

movies in its three locations the next day. The *total number* of movies doesn't change because the columns sum to 1.

If x_n, y_n, z_n are the numbers of movies in locations $1, 2, 3$, respectively, on day n, and $v_n = (x_n, y_n, z_n)$, then:

$$v_n = Av_{n-1} = A^2 v_{n-2} = \cdots = A^n v_0.$$

Recall: This is an example of a **difference equation**.

Red Box probably cares about what v_n is as n gets large: it tells them where the movies will end up *eventually*. This seems to involve computing A^n for large n, but as we will see, they actually only have to compute one eigenvector.

In general: A difference equation $v_{n+1} = Av_n$ is used to model a state change controlled by a matrix:

- v_n is the "state at time n",
- v_{n+1} is the "state at time $n + 1$", and
- $v_{n+1} = Av_n$ means that A is the "change of state matrix."

Fact: 1 is an eigenvalue of a stochastic matrix.

Why? If A is stochastic, then 1 is an eigenvalue of A^T:

$$\begin{pmatrix} .3 & .3 & .4 \\ .4 & .4 & .2 \\ .5 & .3 & .2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Lemma

A and A^T have the same eigenvalues.

Proof: $\det(A - \lambda I) = \det\left((A - \lambda I)^T\right) = \det(A^T - \lambda I)$, so they have the same characteristic polynomial.

Note: This doesn't give a new procedure for finding an eigenvector with eigenvalue 1; it only shows one exists.

Fact: if λ is an eigenvalue of a stochastic matrix, then $|\lambda| \leq 1$. Hence 1 is the *largest* eigenvalue (in absolute value).

Why? If λ is an eigenvalue of A then it is an eigenvalue of A^T.

$$\text{eigenvector } v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \lambda v = A^T v \implies \lambda x_j = \boxed{\sum_{i=1}^n a_{ij} x_i.}$$

*j*th entry of $A^T v$

Choose x_j with the largest absolute value, so $|x_i| \leq |x_j|$ for all i.

$$|\lambda| \cdot |x_j| = \left| \sum_{i=1}^n a_{ij} x_i \right| \leq \sum_{i=1}^n a_{ij} \cdot |x_i| \leq \sum_{i=1}^n a_{ij} \cdot |x_j| = 1 \cdot |x_j|,$$

positive $= \sum_i a_{ij}$

$\geq |x_i|$

so $|\lambda| \leq 1$.

Better fact: if $\lambda \neq 1$ is an eigenvalue of a *positive* stochastic matrix, then $|\lambda| < 1$.

Let $A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$. This is a positive stochastic matrix.

This matrix is diagonalizable:

$$A = CDC^{-1} \quad \text{for} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Let $w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ be the columns of C.

$$A(a_1 w_1 + a_2 w_2) = a_1 w_1 + \frac{1}{2} a_2 w_2$$

$$A^2(a_1 w_1 + a_2 w_2) = a_1 w_1 + \frac{1}{4} a_2 w_2$$

$$A^3(a_1 w_1 + a_2 w_2) = a_1 w_1 + \frac{1}{8} a_2 w_2$$

$$\vdots$$
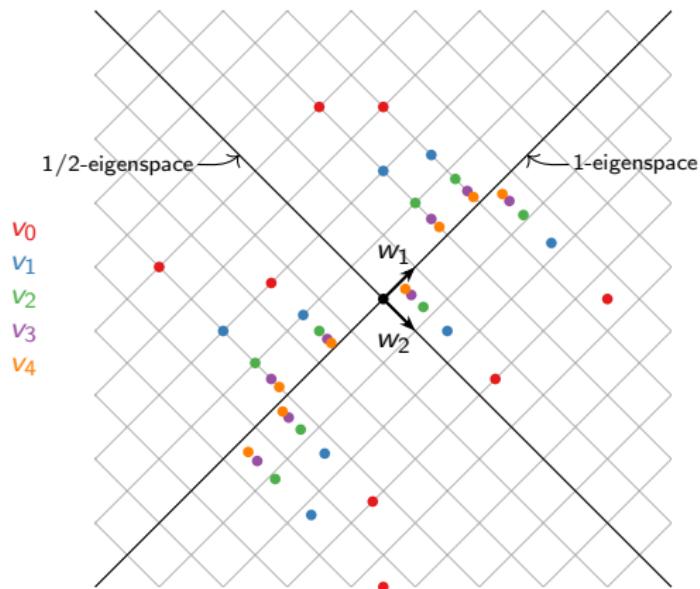
$$A^n(a_1 w_1 + a_2 w_2) = a_1 w_1 + \frac{1}{2^n} a_2 w_2$$

When n is large, the second term disappears, so $A^n x$ approaches $a_1 w_1$, which is an *eigenvector with eigenvalue 1* (assuming $a_1 \neq 0$). So all vectors get "sucked into the 1-eigenspace," which is spanned by $w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Diagonalizable Stochastic Matrices

Example, continued

[interactive]

All vectors get "sucked into the 1-eigenspace."

Diagonalizable Stochastic Matrices

The Red Box matrix $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$ has characteristic polynomial

$$f(\lambda) = -\lambda^3 + 0.12\lambda - 0.02 = -(\lambda - 1)(\lambda + 0.2)(\lambda - 0.1).$$

So 1 is indeed the largest eigenvalue. Since A has 3 distinct eigenvalues, it is diagonalizable:

$$A = C \begin{pmatrix} 1 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & -.2 \end{pmatrix} C^{-1} = CDC^{-1}.$$

Hence it is easy to compute the powers of A:

$$A^n = C \begin{pmatrix} 1 & 0 & 0 \\ 0 & (.1)^n & 0 \\ 0 & 0 & (-.2)^n \end{pmatrix} C^{-1} = CD^n C^{-1}.$$

Let w_1, w_2, w_3 be the columns of C, i.e. the eigenvectors of C with respective eigenvalues $1, .1, -.2$.

Let $a_1 w_1 + a_2 w_2 + a_3 w_3$ be any vector in $\mathbf{R}^3$.

$$A(a_1 w_1 + a_2 w_2 + a_3 w_3) = a_1 w_1 + (.1) a_2 w_2 + (-.2) a_3 w_3$$
$$A^2(a_1 w_1 + a_2 w_2 + a_3 w_3) = a_1 w_1 + (.1)^2 a_2 w_2 + (-.2)^2 a_3 w_3$$
$$A^3(a_1 w_1 + a_2 w_2 + a_3 w_3) = a_1 w_1 + (.1)^3 a_2 w_2 + (-.2)^3 a_3 w_3$$

$$\vdots$$

$$A^n(a_1 w_1 + a_2 w_2 + a_3 w_3) = a_1 w_1 + (.1)^n a_2 w_2 + (-.2)^n a_3 w_3$$

As n becomes large, this approaches $a_1 w_1$, which is an *eigenvector with eigenvalue* 1 (assuming $a_1 \neq 0$). So all vectors get "sucked into the 1-eigenspace," which (I computed) is spanned by
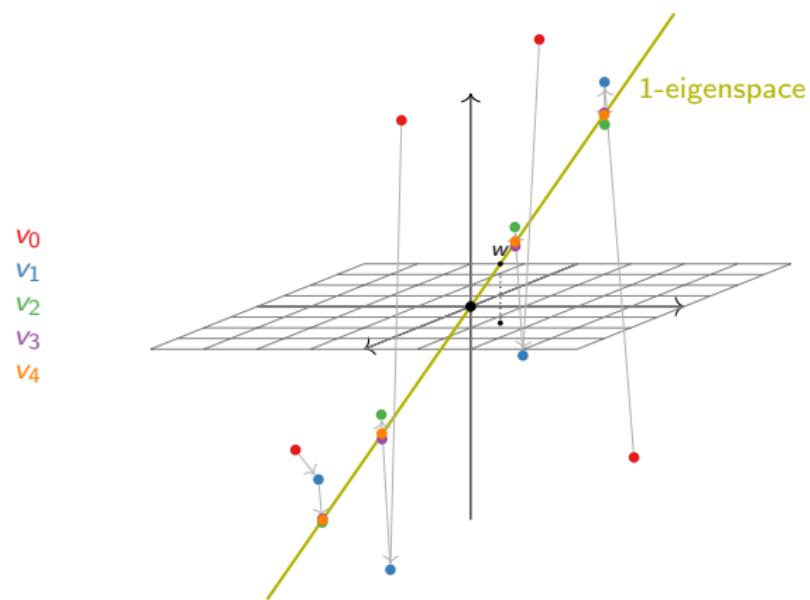
$$w = w_1 = \frac{1}{18} \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix}.$$

(We'll see in a moment why I chose that eigenvector.)

Start with a vector v_0 (the number of movies on the first day), let $v_1 = Av_0$ (the number of movies on the second day), let $v_2 = Av_1$, etc.

We see that v_n approaches an eigenvector with eigenvalue 1 as n gets large: all vectors get "sucked into the 1-eigenspace." [interactive]

If A is the Red Box matrix, and v_n is the vector representing the number of movies in the three locations on day n, then

$$v_{n+1} = Av_n.$$

For any starting distribution v_0 of videos in red boxes, after enough days, the distribution v ($= v_n$ for n large) is an eigenvector with eigenvalue 1:

$$Av = v.$$

In other words, eventually each kiosk has the same number of movies, every day.

Moreover, we know exactly what v is: it is the multiple of $w \sim (0.39, 0.33, 0.28)$ that represents the same number of videos as in v_0. (Remember the total number of videos never changes.)

Presumably, Red Box really does have to do this kind of analysis to determine how many videos to put in each box.

Perron–Frobenius Theorem

Definition
A *steady state* for a stochastic matrix A is an eigenvector w with eigenvalue 1, such that all entries are *positive* and sum to 1.

Perron–Frobenius Theorem
If A is a positive stochastic matrix, then it admits a unique steady state vector w, which spans the 1-eigenspace.

Moreover, for any vector v_0 with entries summing to some number c, the iterates $v_1 = Av_0$, $v_2 = Av_1$, ..., $v_n = Av_{n-1}$, ..., approach cw as n gets large.

Translation: The Perron–Frobenius Theorem says the following:

- The 1-eigenspace of a positive stochastic matrix A is a line.
- To compute the steady state, find any 1-eigenvector (as usual), then divide by the sum of the entries; the resulting vector w has entries that sum to 1, and are *automatically* positive.
- Think of w as a vector of steady state *percentages*: if the movies are distributed according to these percentages today, then they'll be in the same distribution tomorrow.
- The sum c of the entries of v_0 is the total number of movies; eventually, the movies arrange themselves according to the steady state percentage, i.e., $v_n \to cw$.

Steady State
Red Box example

Consider the Red Box matrix $A = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$.

I computed $\text{Nul}(A - I)$ and found that

$$w' = \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix}$$

is an eigenvector with eigenvalue 1.

To get a steady state, I divided by $18 = 7 + 6 + 5$ to get

$$w = \frac{1}{18} \begin{pmatrix} 7 \\ 6 \\ 5 \end{pmatrix} \sim (0.39, 0.33, 0.28).$$

This says that eventually, 39% of the movies will be in location 1, 33% will be in location 2, and 28% will be in location 3, every day.

So if you start with 100 total movies, eventually you'll have $100w = (39, 33, 28)$ movies in the three locations, every day.

The Perron–Frobenius Theorem says that our analysis of the Red Box matrix works for *any* positive stochastic matrix—whether or not it is diagonalizable!

Google's PageRank

Internet searching in the 90's was a pain. Yahoo or AltaVista would scan pages for your search text, and just list the results with the most occurrences of those words.

Not surprisingly, the more unsavory websites soon learned that by putting the words "Alanis Morissette" a million times in their pages, they could show up first every time an angsty teenager tried to find *Jagged Little Pill* on Napster.

Larry Page and Sergey Brin invented a way to rank pages by *importance*. They founded Google based on their algorithm.

Here's how it works. (roughly)

Reference:

`http://www.math.cornell.edu/~mec/Winter2009/RalucaRemus/Lecture3/lecture3.html`

The Importance Rule

Each webpage has an associated importance, or **rank**. This is a positive number.
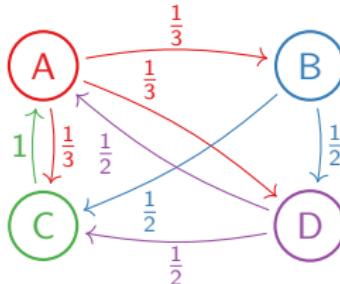
> ### The Importance Rule
>
> If page P links to n other pages $Q_1, Q_2, \ldots, Q_n$, then each Q_i should inherit $\frac{1}{n}$ of P's importance.

- So if a very important page links to your webpage, your webpage is considered important.
- And if a ton of unimportant pages link to your webpage, then it's still important.
- But if only one crappy site links to yours, your page isn't important.

Random surfer interpretation: a "random surfer" just sits at his computer all day, randomly clicking on links. The pages he spends the most time on should be the most important. This turns out to be equivalent to the rank.

The Importance Matrix

Consider the following Internet with only four pages. Links are indicated by arrows.

Page A has 3 links, so it passes $\frac{1}{3}$ of its importance to pages B, C, D.
Page B has 2 links, so it passes $\frac{1}{2}$ of its importance to pages C, D.
Page C has one link, so it passes all of its importance to page A.
Page D has 2 links, so it passes $\frac{1}{2}$ of its importance to pages A, C.

In terms of matrices, if $v = (a, b, c, d)$ is the vector containing the ranks a, b, c, d of the pages A, B, C, D, then

Importance Rule

importance matrix: ij entry is importance page j passes to page i

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c + \frac{1}{2}d \\ \frac{1}{3}a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}d \\ \frac{1}{3}a + \frac{1}{2}b \end{pmatrix} \overset{!}{=} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

The 25 Billion Dollar Eigenvector

Observations:

- The importance matrix is a stochastic matrix! The columns each contain $1/n$ ($n =$ number of links), n times.
- The rank vector is an eigenvector with eigenvalue 1!

Random surfer interpretation: If a random surfer has probability (a, b, c, d) to be on page A, B, C, D, respectively, then after clicking on a random link, the probability he'll be on each page is

$$
\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c + \frac{1}{2}d \\ \frac{1}{3}a \\ \frac{1}{3}a + \frac{1}{2}b + \frac{1}{2}d \\ \frac{1}{3}a + \frac{1}{2}b \end{pmatrix}.
$$

The rank vector is a *steady state* for the importance matrix: it's the probability vector (a, b, c, d) such that, after clicking on a random link, the random surfer will have the *same probability* of being on each page.
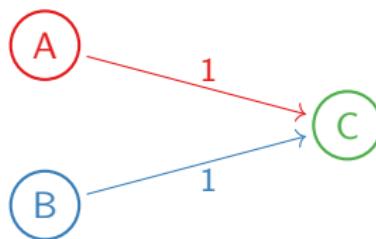
So, the important (high-ranked) pages are those where a random surfer will end up most often.

Observation: the importance matrix is *not* positive: it's only nonnegative. So we can't apply the Perron–Frobenius theorem. Does this cause problems? Yes!

Consider the following Internet:

The importance matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$. This has characteristic polynomial
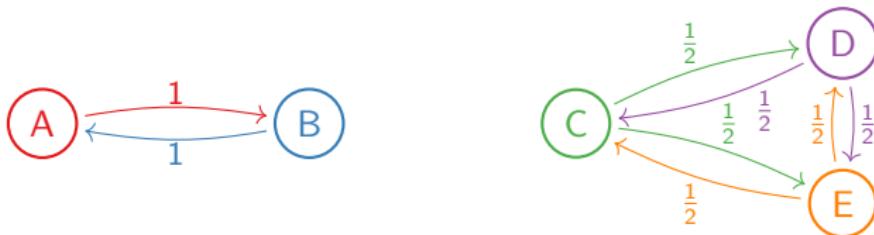
$$f(\lambda) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 1 & 1 & -\lambda \end{pmatrix} = -\lambda^3.$$

So 1 is not an eigenvalue at all: there is no rank vector! (It is not stochastic.)

Problems with the Importance Matrix

Consider the following Internet:

The importance matrix is $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. This has linearly independent

eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, both with eigenvalue 1. So there is more than

one rank vector!

The Google Matrix

Here is Page and Brin's solution. First we fix the importance matrix A as follows: replace a column if zeros with a column of $1/N$s, where N is the number of pages.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \rightsquigarrow \quad A' = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 0 & 1/3 \\ 1 & 1 & 1/3 \end{pmatrix}.$$

The **modified importance matrix** A' is always stochastic.

Now fix p in $(0, 1)$, called the **damping factor**. (A typical value is $p = 0.15$.)
The **Google Matrix** is

$$M = (1 - p) \cdot A' + p \cdot B \quad \text{where} \quad B = \frac{1}{N} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

N is the total number of pages.

In the random surfer interpretation, this matrix M says: with probability p, our surfer will surf to a completely random page; otherwise, he'll click a random link. On a page with no links, he'll always surf to a completely random page.

Lemma
The Google matrix is a positive stochastic matrix.
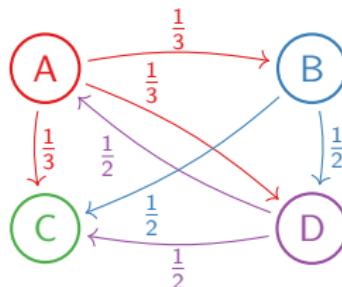
> The PageRank vector is the steady state for the Google Matrix.

This exists and has positive entries by the Perron–Frobenius theorem. The hard part is calculating it: the Google matrix has 1 gazillion rows.

Consider this Internet:

The importance and modified importance matrices are

$$A = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \xrightarrow{\text{modify}} A' = \begin{pmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{4} & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{4} & 0 \end{pmatrix}$$

If we choose the damping factor $p = .15$, then the Google matrix is

$$M = (1-p)A' + pB = \begin{pmatrix} 0.0375 & 0.0375 & 0.2500 & 0.4625 \\ 0.3208 & 0.0375 & 0.2500 & 0.0375 \\ 0.3208 & 0.4625 & 0.2500 & 0.4625 \\ 0.3208 & 0.4625 & 0.2500 & 0.0375 \end{pmatrix}$$
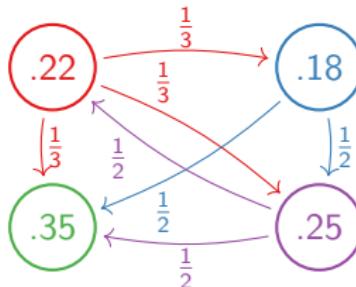
The Google Matrix
Example, Continued

$$M = \begin{pmatrix} 0.0375 & 0.0375 & 0.2500 & 0.4625 \\ 0.3208 & 0.0375 & 0.2500 & 0.0375 \\ 0.3208 & 0.4625 & 0.2500 & 0.4625 \\ 0.3208 & 0.4625 & 0.2500 & 0.0375 \end{pmatrix}$$

Row reduce $M - I$ to find the steady-state vector:

$$v = \begin{pmatrix} 0.2192 \\ 0.1752 \\ 0.3558 \\ 0.2498 \end{pmatrix}$$

This is the PageRank!

Summary

- **Stochastic** and **positive stochastic** matrices model probabilistic systems.
- We care about the long-term behavior of such a system. This is called the **steady state**. It tells us the eventual state of the system.
- The Perron–Frobenius theorem says that a positive stochastic matrix always has a unique steady state.
- If you can understand the RedBox example, then you understand almost everything.
- The Google matrix is an example of a positive stochastic matrix.
- The steady state of the Google matrix is the PageRank.

Chapter 6

Orthogonality

Section 6.1

Dot Products and Orthogonality
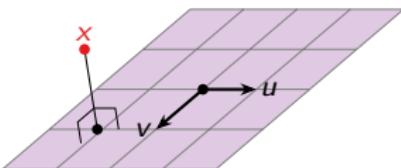
Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
- Solve the matrix equation $Ax = \lambda x$
- Almost solve the equation $Ax = b$

We are now aiming at the last topic.

Idea: In the real world, data is imperfect. Suppose you measure a data point x which you know for theoretical reasons must lie on a plane spanned by two vectors u and v.

Due to measurement error, though, the measured x is not actually in $\text{Span}\{u, v\}$. In other words, the equation $au + bv = x$ has no solution. What do you do? The real value is probably the *closest* point to x on $\text{Span}\{u, v\}$. Which point is that?

The Dot Product

We need a notion of *angle* between two vectors, and in particular, a notion of *orthogonality* (i.e. when two vectors are perpendicular). This is the purpose of the dot product.

Definition

The **dot product** of two vectors x, y in $\mathbf{R}^n$ is

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \overset{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Thinking of x, y as column vectors, this is the same as $x^T y$.

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32.$$

Properties of the Dot Product

Many usual arithmetic rules hold, as long as you remember you can only dot two vectors together, and that the result is a scalar.

- $x \cdot y = y \cdot x$
- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $(cx) \cdot y = c(x \cdot y)$

Dotting a vector with itself is special:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Hence:

- $x \cdot x \geq 0$
- $x \cdot x = 0$ if and only if $x = 0$.

Important: $x \cdot y = 0$ does *not* imply $x = 0$ or $y = 0$. For example, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$.
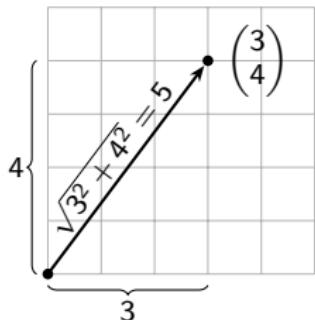
Definition

The **length** or **norm** of a vector x in $\mathbf{R}^n$ is

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Why is this a good definition? The Pythagorean theorem!

$$\left\|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\| = \sqrt{3^2 + 4^2} = 5$$

Fact

If x is a vector and c is a scalar, then $\|cx\| = |c| \cdot \|x\|$.

$$\left\|\begin{pmatrix} 6 \\ 8 \end{pmatrix}\right\| = \left\|2\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\| = 2\left\|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right\| = 10$$

The Dot Product and Distance

Definition

The **distance** between two points x, y in $\mathbf{R}^n$ is
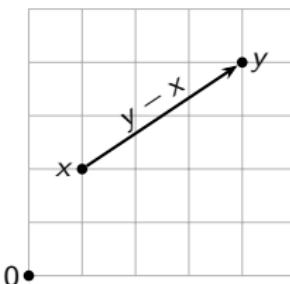
$$\text{dist}(x, y) = \|y - x\|.$$

This is just the length of the vector from x to y.

Example

Let $x = (1, 2)$ and $y = (4, 4)$. Then

$$\text{dist}(x, y) = \|y - x\| = \left\| \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\| = \sqrt{3^2 + 2^2} = \sqrt{13}.$$

Unit Vectors

Definition
A **unit vector** is a vector v with length $\|v\| = 1$.

Example
The unit coordinate vectors are unit vectors:

$$\|e_1\| = \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

Definition
Let x be a nonzero vector in $\mathbf{R}^n$. The **unit vector in the direction of** x is the vector $\dfrac{x}{\|x\|}$.

This is in fact a unit vector:

$$\underset{\text{scalar}}{\left\| \frac{x}{\|x\|} \right\|} = \frac{1}{\|x\|} \|x\| = 1.$$
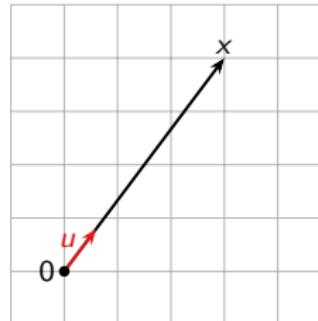
## Example

What is the unit vector in the direction of $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

$$u = \frac{x}{\|x\|} = \frac{1}{\sqrt{3^2 + 4^2}} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$
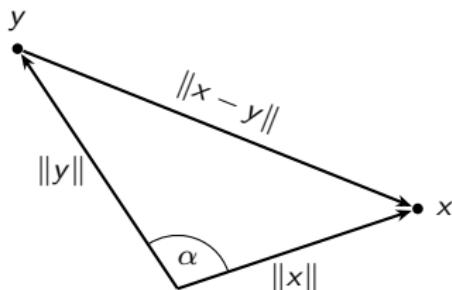
Orthogonality

Why is this a good definition? The Pythagorean theorem / law of cosines!

Law of cosines:
$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\| \, \|y\| \cos \alpha$$
$$\alpha = 90° \iff \cos \alpha = 0$$

x and y are perpendicular $\iff \|x\|^2 + \|y\|^2 = \|x - y\|^2$

$\iff x \cdot x + y \cdot y = (x - y) \cdot (x - y)$

$\iff x \cdot x + y \cdot y = x \cdot x + y \cdot y - 2x \cdot y$

$\iff x \cdot y = 0$

Fact: $x \perp y \iff \|x - y\|^2 = \|x\|^2 + \|y\|^2$

Problem: Find *all* vectors orthogonal to $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

We have to find all vectors x such that $x \cdot v = 0$. This means solving the equation

$$0 = x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3.$$

The parametric form for the solution is $x_1 = -x_2 + x_3$, so the parametric vector form of the general solution is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For instance, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \perp \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ because $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$.

Problem: Find *all* vectors orthogonal to both $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Now we have to solve the system of two homogeneous equations

$$0 = x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = x_1 + x_2 - x_3$$

$$0 = x \cdot w = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x_1 + x_2 + x_3.$$

In matrix form:

The rows are v and w $\longrightarrow$ $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

The parametric vector form of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Problem: Find all vectors orthogonal to some number of vectors $v_1, v_2, \ldots, v_m$ in $\mathbf{R}^n$.

This is the same as finding all vectors x such that

$$0 = v_1^T x = v_2^T x = \cdots = v_m^T x.$$

Putting the *row* vectors $v_1^T, v_2^T, \ldots, v_m^T$ into a matrix, this is the same as finding all x such that

$$\begin{pmatrix} - v_1^T - \\ - v_2^T - \\ \vdots \\ - v_m^T - \end{pmatrix} x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_m \cdot x \end{pmatrix} = 0.$$

> **Important**
>
> The set of all vectors orthogonal to some vectors $v_1, v_2, \ldots, v_m$ in $\mathbf{R}^n$ is the *null space* of the $m \times n$ matrix you get by "turning them sideways and smooshing them together:" $\begin{pmatrix} - v_1^T - \\ - v_2^T - \\ \vdots \\ - v_m^T - \end{pmatrix}$.
>
> In particular, this set is a subspace!

Summary

- The **dot product** of vectors x, y in $\mathbf{R}^n$ is the number $x^T y$.
- The **length** or **norm** of a vector x in $\mathbf{R}^n$ is $\|x\| = \sqrt{x \cdot x}$.
- The **distance** between two vectors x, y in $\mathbf{R}^n$ is $\text{dist}(x, y) = \|y - x\|$.
- A **unit vector** is a vector v with length $\|v\| = 1$.
- The **unit vector in the direction of** x is $x/\|x\|$.
- Two vectors x, y are **orthogonal** if $x \cdot y = 0$.
- The set of all vectors orthogonal to some vectors $v_1, v_2, \ldots, v_m$ in $\mathbf{R}^n$ is the null space of the matrix

$$
\begin{pmatrix}
- v_1^T - \\
- v_2^T - \\
\vdots \\
- v_m^T -
\end{pmatrix}.
$$

Section 6.2

Orthogonal Complements

Orthogonal Complements
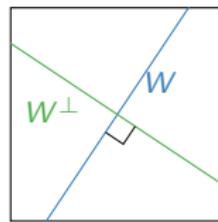
Definition
Let W be a subspace of $\mathbf{R}^n$. Its **orthogonal complement** is

$$W^{\perp} = \left\{ v \text{ in } \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W \right\} \qquad \text{read "W perp".}$$
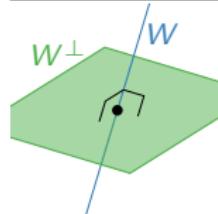
$W^{\perp}$ is orthogonal complement
A^T is transpose

Pictures:
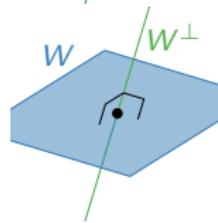
The orthogonal complement of a line in $\mathbf{R}^2$ is the perpendicular line.   [interactive]

The orthogonal complement of a line in $\mathbf{R}^3$ is the perpendicular plane.   [interactive]

The orthogonal complement of a plane in $\mathbf{R}^3$ is the perpendicular line.   [interactive]

Poll

For example, if W is the xy-plane, then $W^\perp$ is the zw-plane:

$$\begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z \\ w \end{pmatrix} = 0.$$

Let W be a subspace of $\mathbf{R}^n$.

Facts:

1. $W^\perp$ is also a subspace of $\mathbf{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $A = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix}$ and $W = \text{Col } A$, then $W^\perp = \text{Nul}(A^T)$ since

$$W^\perp = \text{all vectors orthogonal to each } v_1, v_2, \ldots, v_m$$
$$= \left\{ x \text{ in } \mathbf{R}^n \mid x \cdot v_i = 0 \text{ for all } i = 1, 2, \ldots, m \right\}$$
$$= \text{Nul} \begin{pmatrix} -v_1^T- \\ -v_2^T- \\ \vdots \\ -v_m^T- \end{pmatrix} = \text{Nul}(A^T).$$

Let's check 1.

- Is 0 in $W^\perp$? Yes: $0 \cdot w = 0$ for any w in W.
- Suppose x, y are in $W^\perp$. So $x \cdot w = 0$ and $y \cdot w = 0$ for all w in W. Then $(x + y) \cdot w = x \cdot w + y \cdot w = 0 + 0 = 0$ for all w in W. So $x + y$ is also in $W^\perp$.
- Suppose x is in $W^\perp$. So $x \cdot w = 0$ for all w in W. If c is a scalar, then $(cx) \cdot w = c(x \cdot 0) = c(0) = 0$ for any w in W. So cx is in $W^\perp$.

Problem: if $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, compute $W^\perp$.

By property 4, we have to find the null space of the matrix whose rows are $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, which we did before:

$$\text{Nul} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

[interactive]

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul} \begin{pmatrix} - v_1^T - \\ - v_2^T - \\ \vdots \\ - v_m^T - \end{pmatrix}$$

Definition

The **row space** of an $m \times n$ matrix A is the span of the *rows* of A. It is denoted Row A. Equivalently, it is the column space of A^T:

$$\text{Row } A = \text{Col } A^T.$$

It is a subspace of $\mathbf{R}^n$.

We showed before that if A has rows $v_1^T, v_2^T, \ldots, v_m^T$, then

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul } A.$$

Hence we have shown:

Fact: $(\text{Row } A)^\perp = \text{Nul } A$.

Replacing A by A^T, and remembering Row $A^T = \text{Col } A$:

Fact: $(\text{Col } A)^\perp = \text{Nul } A^T$.

Using property 2 and taking the orthogonal complements of both sides, we get:

Fact: $(\text{Nul } A)^\perp = \text{Row } A$ and $\text{Col } A = (\text{Nul } A^T)^\perp$.

Even though $\text{Row}(A)$ lives in $\mathbf{R}^n$ and $\text{Col}(A)$ lives in $\mathbf{R}^m$ if A is an $m \times n$ matrix, both subspaces have the same dimension.

Theorem

If A is an $m \times n$ matrix, then $\dim(\text{Row } A) = \dim(\text{Col } A)$.

Orthogonal Complements of Most of the Subspaces We've Seen

For any vectors $v_1, v_2, \ldots, v_m$:

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul} \begin{pmatrix} - v_1^T - \\ - v_2^T - \\ \vdots \\ - v_m^T - \end{pmatrix}$$

For any matrix A:

$$\text{Row } A = \text{Col } A^T$$

and

$$(\text{Row } A)^\perp = \text{Nul } A \qquad \text{Row } A = (\text{Nul } A)^\perp$$
$$(\text{Col } A)^\perp = \text{Nul } A^T \qquad \text{Col } A = (\text{Nul } A^T)^\perp$$
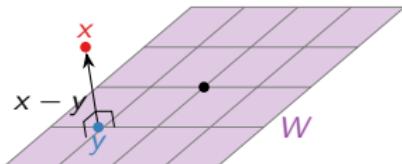
For any other subspace W, first find a basis $v_1, \ldots, v_m$, then use the above trick to compute $W^\perp = \text{Span}\{v_1, \ldots, v_m\}^\perp$.

Section 6.3

Orthogonal Projections (will finish in next set of slides)

Best Approximation

Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W.

Due to measurement error, though, the measured x is not actually in W. Best approximation: y is the *closest* point to x on W.

How do you know that y is the closest point? The vector from y to x is orthogonal to W: it is in the *orthogonal complement* $W^\perp$.

Theorem

Every vector x in $\mathbf{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.
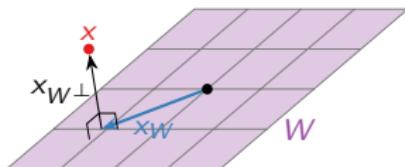
The equation $x = x_W + x_{W^\perp}$ is called the **orthogonal decomposition** of x (with respect to W).

The vector x_W is the **orthogonal projection** of x onto W.

The vector x_W *is the closest vector to x on W.*

[interactive 1]   [interactive 2]

Theorem

Every vector x in $\mathbf{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

Why?

Uniqueness: suppose $x = x_W + x_{W^\perp} = x'_W + x'_{W^\perp}$ for x_W, x'_W in W and $x_{W^\perp}, x'_{W^\perp}$ in $W^\perp$. Rewrite:

$$x_W - x'_W = x'_{W^\perp} - x_{W^\perp}.$$

The left side is in W, and the right side is in $W^\perp$, so they are both in $W \cap W^\perp$. But the only vector that is perpendicular to itself is the zero vector! Hence

$$0 = x_W - x'_W \implies x_W = x'_W$$
$$0 = x_{W^\perp} - x'_{W^\perp} \implies x_{W^\perp} = x'_{W^\perp}$$

Existence: We will compute the orthogonal decomposition later using orthogonal projections.
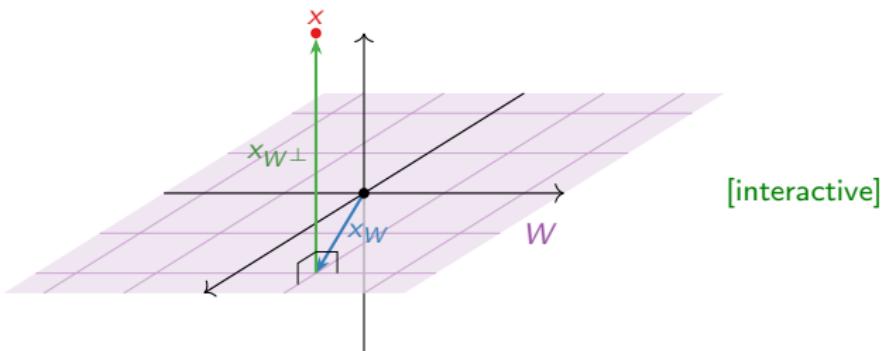
Orthogonal Decomposition
Example

Let W be the xy-plane in $\mathbf{R}^3$. Then $W^\perp$ is the z-axis.

$$x = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \implies x_W = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad x_{W^\perp} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \implies x_W = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \qquad x_{W^\perp} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}.$$

This is just decomposing a vector into a "horizontal" component (in the xy-plane) and a "vertical" component (on the z-axis).

[interactive]

Orthogonal Decomposition

Problem: Given x and W, how do you compute the decomposition $x = x_W + x_{W^\perp}$?

Observation: It is enough to compute x_W, because $x_{W^\perp} = x - x_W$.

Theorem (The $A^T A$ Trick)

Let W be a subspace of $\mathbf{R}^n$, let $v_1, v_2, \ldots, v_m$ be a spanning set for W (e.g., a basis), and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then for any x in $\mathbf{R}^n$, the matrix equation

$$A^T A v = A^T x \quad \text{(in the unknown vector } v)$$

is consistent, and $x_W = Av$ for any solution v.

Recipe for Computing $x = x_W + x_{W^\perp}$

- Write W as a column space of a matrix A.
- Find a solution v of $A^T A v = A^T x$ (by row reducing).
- Then $x_W = Av$ and $x_{W^\perp} = x - x_W$.

The $A^T A$ Trick

Example

Problem: Compute the orthogonal projection of a vector $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$ onto the xy-plane.

First we need a basis for the xy-plane: let's choose

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \rightsquigarrow \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$A^T A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 \qquad A^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then $A^T A v = v$ and $A^T x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, so the only solution of $A^T A v = A^T x$ is $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Therefore,

$$x_W = A v = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$

Problem: Let

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}.$$

Compute the distance from x to W.

The distance from x to W is $\|x_{W^\perp}\|$, so we need to compute the orthogonal projection. First we need a basis for $W = \text{Nul} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$. This matrix is in RREF, so the parametric form of the solution set is

$$\begin{array}{l} x_1 = x_2 - x_3 \\ x_2 = x_2 \\ x_3 = x_3 \end{array} \quad \overset{\text{PVF}}{\rightsquigarrow} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence we can take a basis to be

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \rightsquigarrow \quad A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Problem: Let

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}.$$

Compute the distance from x to W.

We compute

$$A^T A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A^T x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

To solve $A^T A v = A^T x$ we form an augmented matrix and row reduce:

$$\begin{pmatrix} 2 & -1 & | & 3 \\ -1 & 2 & | & 2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 1 & 7/3 \end{pmatrix} \rightsquigarrow v = \frac{1}{3} \begin{pmatrix} 8 \\ 7 \end{pmatrix}.$$

$$x_W = Av = \frac{1}{3} \begin{pmatrix} 1 \\ 8 \\ 7 \end{pmatrix} \qquad x_{W^\perp} = x - x_W = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}.$$

The distance is $\|x_{W^\perp}\| = \frac{1}{3}\sqrt{4 + 4 + 4} \approx 1.155$.

[interactive]

Theorem (The $A^T A$ Trick)

Let W be a subspace of $\mathbf{R}^n$, let $v_1, v_2, \ldots, v_m$ be a spanning set for W (e.g., a basis), and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then for any x in $\mathbf{R}^n$, the matrix equation

$$A^T A v = A^T x \quad \text{(in the unknown vector } v\text{)}$$

is consistent, and $x_W = Av$ for any solution v.

Proof: Let $x = x_W + x_{W^\perp}$. Then $x_{W^\perp}$ is in $W^\perp = \text{Nul}(A^T)$, so $A^T x_{W^\perp} = 0$. Hence

$$A^T x = A^T (x_W + x_{W^\perp}) = A^T x_W + A^T x_{W^\perp} = A^T x_W.$$

Since x_W is in $W = \text{Span}\{v_1, v_2, \ldots, v_m\}$, we can write

$$x_W = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$

If $v = (c_1, c_2, \ldots, c_m)$ then $Av = x_W$, so

$$A^T x = A^T x_W = A^T Av.$$

Orthogonal Projection onto a Line

Problem: Let $L = \mathrm{Span}\{u\}$ be a line in $\mathbf{R}^n$ and let x be a vector in $\mathbf{R}^n$. Compute x_L.
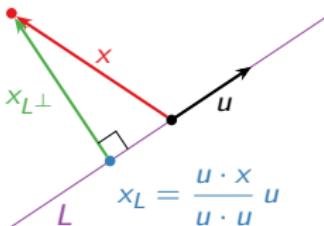
We have to solve $u^T u v = u^T x$, where u is an $n \times 1$ matrix. But $u^T u = u \cdot u$ and $u^T x = u \cdot x$ are scalars, so

$$v = \frac{u \cdot x}{u \cdot u} \quad \implies \quad x_L = uv = \frac{u \cdot x}{u \cdot u} u.$$

Projection onto a Line

The projection of x onto a line $L = \mathrm{Span}\{u\}$ is

$$x_L = \frac{u \cdot x}{u \cdot u} u \qquad x_{L\perp} = x - x_L.$$

Orthogonal Projection onto a Line
Example
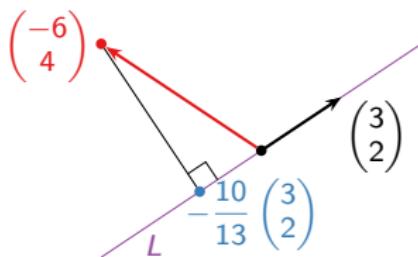
Problem: Compute the orthogonal projection of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line L spanned by $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and find the distance from u to L.

$$x_L = \frac{x \cdot u}{u \cdot u} u = \frac{-18 + 8}{9 + 4} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad x_{L^\perp} = x - x_L = \frac{1}{13} \begin{pmatrix} -48 \\ 72 \end{pmatrix}.$$

The distance from x to L is

$$\|x_{L^\perp}\| = \frac{1}{13} \sqrt{48^2 + 72^2} \approx 6.656.$$

[interactive]

Summary

Let W be a subspace of $\mathbf{R}^n$.
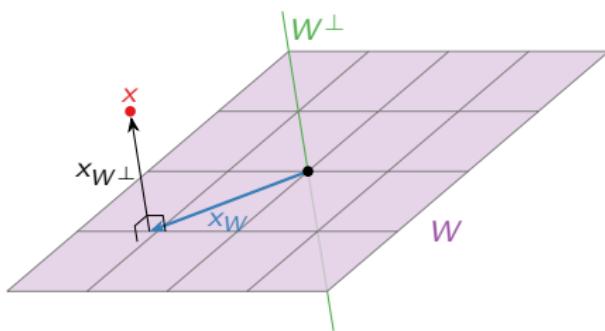
- The **orthogonal complement** $W^\perp$ is the set of all vectors orthogonal to everything in W.
- We have $(W^\perp)^\perp = W$ and $\dim W + \dim W^\perp = n$.
- $\text{Row } A = \text{Col } A^T$, $(\text{Row } A)^\perp = \text{Nul } A$, $\text{Row } A = (\text{Nul } A)^\perp$, $(\text{Col } A)^\perp = \text{Nul } A^T$, $\text{Col } A = (\text{Nul } A^T)^\perp$.
- **Orthogonal decomposition:** any vector x in $\mathbf{R}^n$ can be written in a unique way as $x = x_W + x_{W^\perp}$ for x_W in W and $x_{W^\perp}$ in $W^\perp$. The vector x_W is the **orthogonal projection** of x onto W.
- The vector x_W is the *closest point to x in W*: it is the *best approximation*.
- The *distance* from x to W is $\|x_{W^\perp}\|$.
- If $W = \text{Col } A$ then to compute x_W, solve the equation $A^T A v = A^T x$; then $x_W = Av$.
- If $W = L = \text{Span}\{u\}$ is a line then $x_L = \frac{u \cdot x}{u \cdot u}\, u$.

Recall: Let W be a subspace of $\mathbf{R}^n$.

- The **orthogonal complement** $W^\perp$ is the set of vectors orthogonal to everything in W.
- The **orthogonal decomposition** of a vector x with respect to W is the unique way of writing $x = x_W + x_{W^\perp}$ for x_W in W and $x_{W^\perp}$ in $W^\perp$.
- The vector x_W is the **orthogonal projection** of x onto W. It is the closest vector to x in W.
- To compute x_W, write W as $\operatorname{Col} A$ and solve $A^T A v = A^T x$; then $x_W = Av$.

Projection as a Transformation

let us consider orthogonal projection as a *transformation*.

Definition

Let W be a subspace of $\mathbf{R}^n$. Define a transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n \quad \text{by} \quad T(x) = x_W.$$

This transformation is also called **orthogonal projection** with respect to W.

Theorem

Let W be a subspace of $\mathbf{R}^n$ and let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection with respect to W. Then:

1. T is a *linear* transformation.
2. For every x in $\mathbf{R}^n$, $T(x)$ is the *closest* vector to x in W.
3. For every x in W, we have $T(x) = x$.
4. For every x in $W^\perp$, we have $T(x) = 0$.
5. $T \circ T = T$.
6. The range of T is W and the null space of T is $W^\perp$.

Let W be a subspace of $\mathbf{R}^n$ and let $T \colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection with respect to W.

Since T is a linear transformation, it has a matrix. How do you compute it?

The same as any other linear transformation: compute $T(e_1), T(e_2), \ldots, T(e_n)$.

Problem: Let $L = \text{Span}\left\{\binom{3}{2}\right\}$ and let $T \colon \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection onto L. Compute the matrix A for T.

It's easy to compute orthogonal projection onto a line:

$$
\left.\begin{aligned}
T(e_1) &= (e_1)_L = \frac{u \cdot e_1}{u \cdot u}\, u = \frac{3}{13}\binom{3}{2} \\[2mm]
T(e_2) &= (e_2)_L = \frac{u \cdot e_2}{u \cdot u}\, u = \frac{2}{13}\binom{3}{2}
\end{aligned}\right\}
\quad \Longrightarrow \quad
A = \frac{1}{13}\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}.
$$

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}$$

and let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto W. Compute the matrix B for T.

In the slides for the last lecture we computed $W = \operatorname{Col} A$ for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To compute $T(e_i)$ we have to solve the matrix equation $A^T A v = A^T e_i$. We have

$$A^T A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A^T e_i = \text{the } i\text{th column of } A^T = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}$$

and let $T \colon \mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto W. Compute the matrix B for T.

$$\begin{pmatrix} 2 & -1 & | & 1 \\ -1 & 2 & | & -1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & | & 1/3 \\ 0 & 1 & | & -1/3 \end{pmatrix} \implies T(e_1) = \frac{1}{3} A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & | & 1 \\ -1 & 2 & | & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & | & 2/3 \\ 0 & 1 & | & 1/3 \end{pmatrix} \implies T(e_2) = \frac{1}{3} A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & | & 0 \\ -1 & 2 & | & 1 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & | & 1/3 \\ 0 & 1 & | & 2/3 \end{pmatrix} \implies T(e_2) = \frac{1}{3} A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\implies B = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Theorem

Let $\{v_1, v_2, \ldots, v_m\}$ be a *linearly independent* set in $\mathbf{R}^n$, and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then the $m \times m$ matrix $A^T A$ is invertible.

Proof: We'll show $\mathrm{Nul}(A^T A) = \{0\}$. Suppose $A^T A v = 0$. Then Av is in $\mathrm{Nul}(A^T) = \mathrm{Col}(A)^\perp$. But Av is in $\mathrm{Col}(A)$ as well, so $Av = 0$, and hence $v = 0$ because the columns of A are linearly independent.

Theorem

Let $\{v_1, v_2, \ldots, v_m\}$ be a *linearly independent* set in $\mathbf{R}^n$, and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then the $m \times m$ matrix $A^T A$ is invertible.

Let W be a subspace of $\mathbf{R}^n$ and let $T \colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection with respect to W. Let $\{v_1, v_2, \ldots, v_m\}$ be a *basis* for W and let A be the matrix with columns $v_1, v_2, \ldots, v_m$. To compute $T(x) = x_W$ you solve $A^T A v = A x$; then $x_W = A v$.

$$v = (A^T A)^{-1}(A^T x) \implies T(x) = A v = \left[A(A^T A)^{-1} A^T \right] x.$$

If the columns of A are a *basis* for W then the matrix for T is

$$A(A^T A)^{-1} A^T.$$

Problem: Let $L = \text{Span}\left\{\binom{3}{2}\right\}$ and let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be the orthogonal projection onto L. Compute the matrix A for T.

The set $\left\{\binom{3}{2}\right\}$ is a basis for L, so

$$A = u(u^T u)^{-1} u^T = \frac{1}{u \cdot u} uu^T = \frac{1}{13}\binom{3}{2}\begin{pmatrix} 3 & 2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}.$$

Matrix of Projection onto a Line

If $L = \text{Span}\{u\}$ is a line in $\mathbf{R}^n$, then the matrix for projection onto L is

$$\frac{1}{u \cdot u} uu^T.$$

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}$$

and let $T \colon \mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto W. Compute the matrix B for T.

In the slides for the last lecture we computed $W = \operatorname{Col} A$ for

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The columns are linearly independent, so they form a basis for W. Hence

$$B = A(A^T A)^{-1} A^T = A \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} A^T = \frac{1}{3} A \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} A^T$$

$$= \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Poll

Let W be a subspace of $\mathbf{R}^n$ which is neither the zero subspace nor all of $\mathbf{R}^n$.

The 1-eigenspace is W.

The 0-eigenspace is $W^{\perp}$.

We have $\dim W + \dim W^{\perp} = n$, so that gives n linearly independent eigenvectors already.

So the answer is D.

Theorem

Let W be an m-dimensional subspace of $\mathbf{R}^n$, let $T: \mathbf{R}^n \to W$ be the projection, and let A be the matrix for T. Then:

1. $\operatorname{Col} A = W$, which is the 1-eigenspace.
2. $\operatorname{Nul} A = W^\perp$, which is the 0-eigenspace.
3. $A^2 = A$.
4. A is similar to the diagonal matrix with m ones and $n - m$ zeros on the diagonal.

Proof of 4: Let $v_1, v_2, \ldots, v_m$ be a basis for W, and let $v_{m+1}, v_{m+2}, \ldots, v_n$ be a basis for $W^\perp$. These are (linearly independent) eigenvectors with eigenvalues 1 and 0, respectively, and they form a basis for $\mathbf{R}^n$ because there are n of them.

Example: If W is a plane in $\mathbf{R}^3$, then A is similar to projection onto the xy-plane:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let W be an m-dimensional subspace of $\mathbf{R}^n$, let $T \colon \mathbf{R}^n \to \mathbf{R}^n$ be the orthogonal projection onto W, and let A be the matrix for T. Here's how to diagonalize A:

- Find a basis $\{v_1, v_2, \ldots, v_m\}$ for W.
- Find a basis $\{v_{m+1}, v_{m+2}, \ldots, v_n\}$ for $W^\perp$.
- Then

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{m \text{ ones, } n-m \text{ zeros}} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

Remark: If you already have a basis for W, then it's faster to compute $A(A^T A)^{-1} A^T$.

A Projection Matrix is Diagonalizable

Problem: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \mid x_1 - x_2 + x_3 = 0 \right\}$$

and let $T \colon \mathbf{R}^3 \to \mathbf{R}^3$ be orthogonal projection onto W. Compute the matrix B for T.

As we have seen several times, a basis for W is

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

By definition, W is the orthogonal complement of the line spanned by $(1, -1, 1)$, so $W^\perp = \text{Span}\{(1, -1, 1)\}$. Hence

$$B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Let W be a subspace of $\mathbf{R}^n$ and let x be a vector in $\mathbf{R}^n$.
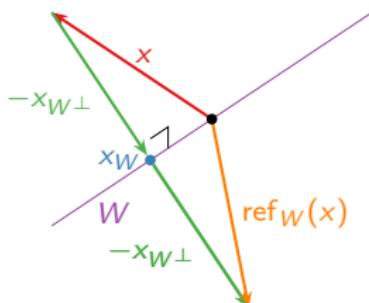
Definition
The **reflection** of x over W is the vector $\text{ref}_W(x) = x - 2x_{W^\perp}$.

In other words, to find $\text{ref}_W(x)$ one starts at x, then moves to $x - x_{W^\perp} = x_W$, then continues in the same direction one more time, to end on the opposite side of W.

Since $x_{W^\perp} = x - x_W$ we have

$$\text{ref}_W(x) = x - 2(x - x_W) = 2x_W - x.$$

If T is the orthogonal projection, then

$$\text{ref}_W(x) = 2T(x) - x.$$

Theorem

Let W be an m-dimensional subspace of $\mathbf{R}^n$, and let A be the matrix for ref_W. Then

1. $\mathrm{ref}_W \circ \mathrm{ref}_W$ is the identity transformation and A^2 is the identity matrix.

2. ref_W and A are invertible; they are their own inverses.

3. The 1-eigenspace of A is W and the -1-eigenspace of A is $W^{\perp}$.

4. A is similar to the diagonal matrix with m ones and $n - m$ negative ones on the diagonal.

5. If B is the matrix for the orthogonal projection onto W, then $A = 2B - I_n$.

Example: Let

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}.$$

The matrix for ref_W is

$$A = 2 \cdot \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} - I_3 = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

Summary

Today we considered orthogonal projection as a transformation.

- Orthogonal projection is a linear transformation.
- We gave three methods to compute its matrix.
- Four if you count the special case when W is a line.
- The matrix for projection onto W has eigenvalues 1 and 0 with eigenspaces W and $W^{\perp}$.
- A projection matrix is diagonalizable.
- Reflection is $2\times$projection minus the identity.

Section 6.5

The Method of Least Squares

Motivation

We now are in a position to solve the motivating problem of this third part of the course:

> **Problem**
>
> Suppose that $Ax = b$ does not have a solution. What is the best possible approximate solution?

To say $Ax = b$ does not have a solution means that b is not in Col A.

The closest possible $\widehat{b}$ for which $Ax = \widehat{b}$ does have a solution is $\widehat{b} = b_{\text{Col } A}$.

Then $A\widehat{x} = \widehat{b}$ is a consistent equation.

A solution $\widehat{x}$ to $A\widehat{x} = \widehat{b}$ is a **least squares solution.**

Least Squares Solutions

Let A be an $m \times n$ matrix.
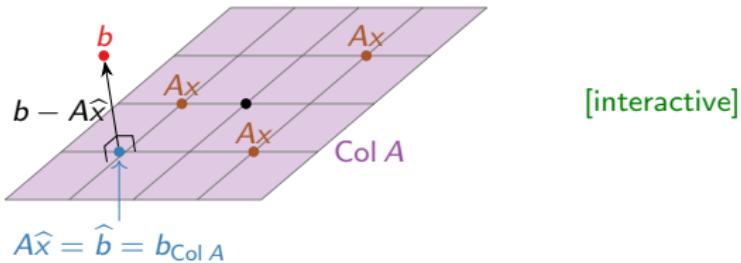
Definition
A **least squares solution** of $Ax = b$ is a vector $\widehat{x}$ in $\mathbf{R}^n$ such that

$$\|b - A\widehat{x}\| \leq \|b - Ax\|$$

for all x in $\mathbf{R}^n$.

Note that $b - A\widehat{x}$ is in $(\text{Col } A)^{\perp}$.

$b - A\widehat{x}$

$A\widehat{x} = \widehat{b} = b_{\text{Col } A}$

Col A

[interactive]

In other words, a least squares solution $\widehat{x}$ solves $Ax = b$ *as closely as possible*.

Equivalently, a least squares solution to $Ax = b$ is a vector $\widehat{x}$ in $\mathbf{R}^n$ such that

$$A\widehat{x} = \widehat{b} = b_{\text{Col } A}.$$

This is because $\widehat{b}$ is the closest vector to b such that $A\widehat{x} = \widehat{b}$ is consistent.

We want to solve $A\widehat{x} = \widehat{b} = b_{\text{Col }A}$. Or, $A\widehat{x} = b_W$ for $W = \text{Col }A$.

To compute b_W we need to solve $A^T A v = A^T b$; then $b_W = Av$.

Conclusion: $\widehat{x}$ is just a solution of $A^T A v = A^T b$!

Theorem

The least squares solutions of $Ax = b$ are the solutions of

$$(A^T A)\widehat{x} = A^T b.$$

Note we compute $\widehat{x}$ directly, without computing $\widehat{b}$ first.

Find the least squares solutions of $Ax = b$ where:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.$$

We have

$$A^T A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$$

and

$$A^T b = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}.$$

Row reduce:

$$\begin{pmatrix} 5 & 3 & | & 0 \\ 3 & 3 & | & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & | & -3 \\ 0 & 1 & | & 5 \end{pmatrix}.$$

So the only least squares solution is $\widehat{x} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

Least Squares Solutions
Example, continued
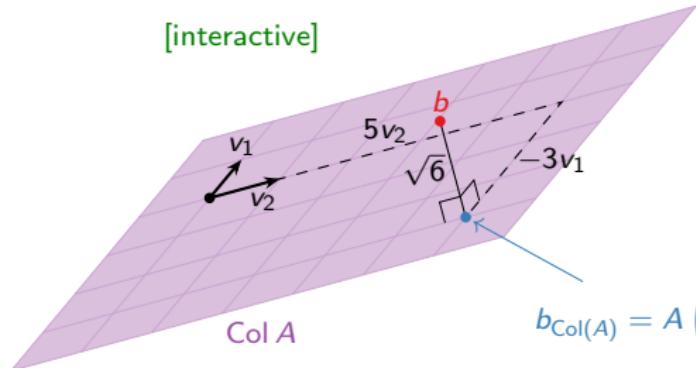
How close did we get?

$$\widehat{b} = A\widehat{x} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

The distance from b is

$$\|b - A\widehat{x}\| = \left\| \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}.$$

[interactive]

Note that

$$\begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

records the coefficients of v_1 and v_2 in $\widehat{b}$.

Find the least squares solutions of $Ax = b$ where:

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

We have

$$A^T A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$$

and

$$A^T b = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

Row reduce:

$$\left(\begin{array}{cc|c} 5 & -1 & 2 \\ -1 & 5 & -2 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -1/3 \end{array} \right).$$

So the only least squares solution is $\widehat{x} = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}.$ [interactive]

When does $Ax = b$ have a *unique* least squares solution $\widehat{x}$?
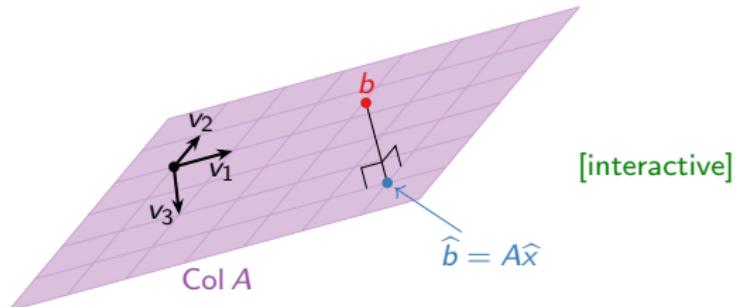
Theorem

Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a *unique* least squares solution for all b in $\mathbf{R}^m$.
2. The columns of A are linearly independent.
3. $A^T A$ is invertible.

In this case, the least squares solution is $(A^T A)^{-1}(A^T b)$.

Why? If the columns of A are linearly *de*pendent, then $A\widehat{x} = \widehat{b}$ has many solutions:

[interactive]

Col A

Note: $A^T A$ is always a square matrix, but it need not be invertible.

Application
Data modeling: best fit line

Find the best fit line through $(0, 6)$, $(1, 0)$, and $(2, 0)$.

The general equation of a line is

$$y = C + Dx.$$

So we want to solve:
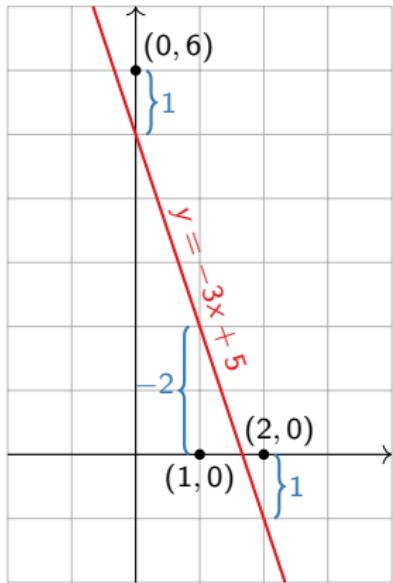
$$6 = C + D \cdot 0$$
$$0 = C + D \cdot 1$$
$$0 = C + D \cdot 2.$$

In matrix form:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.$$

We already saw: the least squares solution is $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$. So the best fit line is

$$y = -3x + 5.$$

$$A \begin{pmatrix} 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Poll

Poll

What does the best fit line minimize?

A. The sum of the squares of the distances from the data points to the line.

B. The sum of the squares of the vertical distances from the data points to the line.

C. The sum of the squares of the horizontal distances from the data points to the line.

D. The maximal distance from the data points to the line.

Answer: B. See the picture on the previous slide.

Find the best fit ellipse for the points $(0, 2)$, $(2, 1)$, $(1, -1)$, $(-1, -2)$, $(-3, 1)$, $(-1, -1)$.

The general equation for an ellipse is

$$x^2 + Ay^2 + Bxy + Cx + Dy + E = 0$$

So we want to solve:

$$
\begin{array}{rrrrrrl}
(0)^2 + & A(2)^2 + & B(0)(2) + & C(0) + & D(2) + E & = 0 \\
(2)^2 + & A(1)^2 + & B(2)(1) + & C(2) + & D(1) + E & = 0 \\
(1)^2 + & A(-1)^2 + & B(1)(-1) + & C(1) + D(-1) + E & = 0 \\
(-1)^2 + & A(-2)^2 + & B(-1)(-2) + & C(-1) + D(-2) + E & = 0 \\
(-3)^2 + & A(1)^2 + & B(-3)(1) + & C(-3) + & D(1) + E & = 0 \\
(-1)^2 + & A(-1)^2 + & B(-1)(-1) + & C(-1) + D(-1) + E & = 0
\end{array}
$$

In matrix form:

$$
\begin{pmatrix}
4 & 0 & 0 & 2 & 1 \\
1 & 2 & 2 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
4 & 2 & -1 & -2 & 1 \\
1 & -3 & -3 & 1 & 1 \\
1 & 1 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
A \\ B \\ C \\ D \\ E
\end{pmatrix}
=
\begin{pmatrix}
0 \\ -4 \\ -1 \\ -1 \\ -9 \\ -1
\end{pmatrix}.
$$

$$A = \begin{pmatrix} 4 & 0 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 4 & 2 & -1 & -2 & 1 \\ 1 & -3 & -3 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ -4 \\ -1 \\ -1 \\ -9 \\ -1 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} 36 & 7 & -5 & 0 & 12 \\ 7 & 19 & 9 & -5 & 1 \\ -5 & 9 & 16 & 1 & -2 \\ 0 & -5 & 1 & 12 & 0 \\ 12 & 1 & -2 & 0 & 6 \end{pmatrix} \qquad A^T b = \begin{pmatrix} -19 \\ 17 \\ 20 \\ -9 \\ -16 \end{pmatrix}$$

Row reduce:

$$\left(\begin{array}{ccccc|c} 36 & 7 & -5 & 0 & 12 & -19 \\ 7 & 19 & 9 & -5 & 1 & 17 \\ -5 & 9 & 16 & 1 & -2 & 20 \\ 0 & -5 & 1 & 12 & 0 & -9 \\ 12 & 1 & -2 & 0 & 6 & -16 \end{array} \right) \rightsquigarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 405/266 \\ 0 & 1 & 0 & 0 & 0 & -89/133 \\ 0 & 0 & 1 & 0 & 0 & 201/133 \\ 0 & 0 & 0 & 1 & 0 & -123/266 \\ 0 & 0 & 0 & 0 & 1 & -687/133 \end{array} \right)$$

Best fit ellipse:
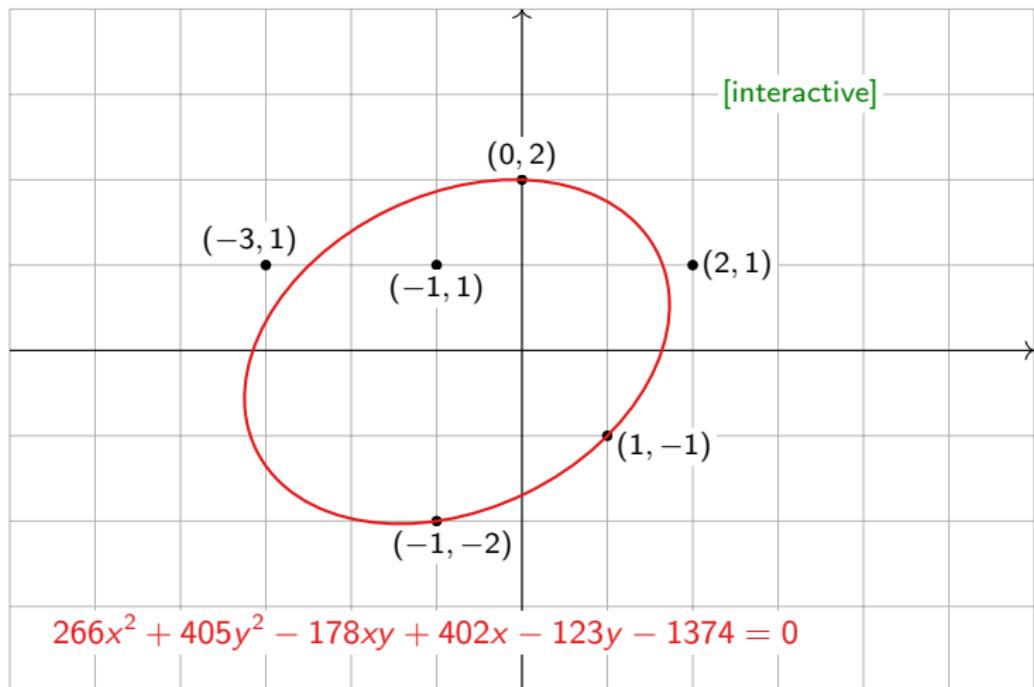
$$x^2 + \frac{405}{266}y^2 - \frac{89}{133}xy + \frac{201}{133}x - \frac{123}{266}y - \frac{687}{133} = 0$$

or

$$266x^2 + 405y^2 - 178xy + 402x - 123y - 1374 = 0.$$

Application
Best fit ellipse, picture

[interactive]

$(0, 2)$

$(-3, 1)$

$(-1, 1)$

$(2, 1)$

$(1, -1)$

$(-1, -2)$

$$266x^2 + 405y^2 - 178xy + 402x - 123y - 1374 = 0$$

Remark: Gauss invented the method of least squares to do exactly this: he predicted the (elliptical) orbit of the asteroid Ceres as it passed behind the sun in 1801.

What least squares problem $Ax = b$ finds the best parabola through the points $(-1, 0.5)$, $(1, -1)$, $(2, -0.5)$, $(3, 2)$?

The general equation for a parabola is

$$y = Ax^2 + Bx + C.$$

So we want to solve:

$$
\begin{aligned}
0.5 &= A(-1)^2 + B(-1) + C \\
-1 &= A(1)^2 + B(1) + C \\
-0.5 &= A(2)^2 + B(2) + C \\
2 &= A(3)^2 + B(3) + C
\end{aligned}
$$

In matrix form:
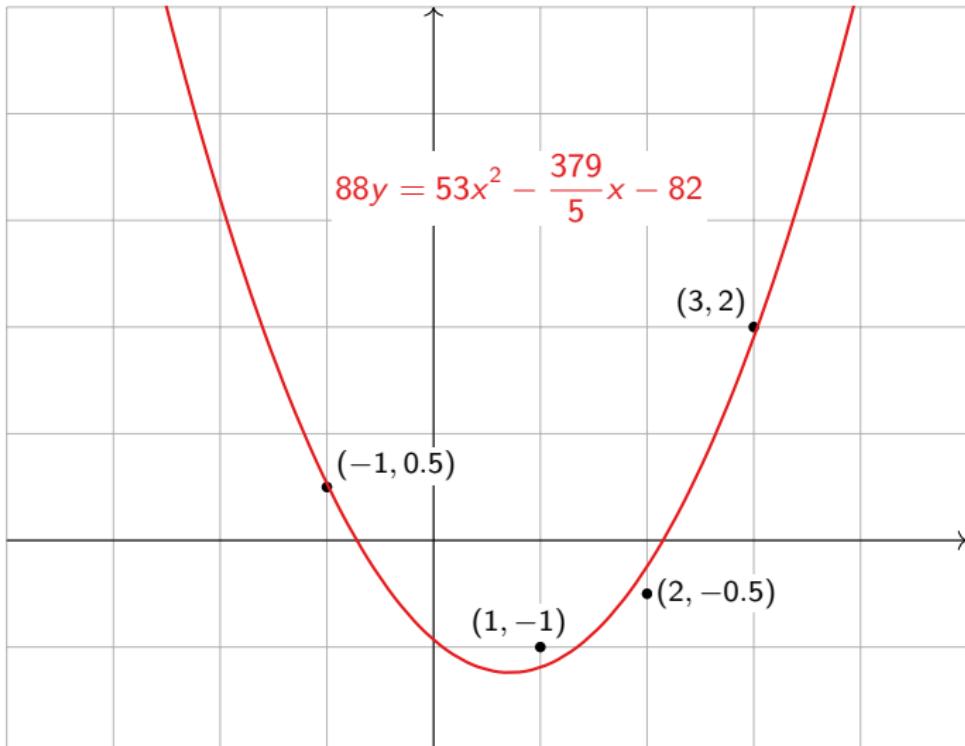
$$
\begin{pmatrix}
1 & -1 & 1 \\
1 & 1 & 1 \\
4 & 2 & 1 \\
9 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix}
=
\begin{pmatrix}
0.5 \\
-1 \\
-0.5 \\
2
\end{pmatrix}.
$$

Answer:

$$88y = 53x^2 - \frac{379}{5}x - 82$$

Application
Best fit parabola, picture

$$88y = 53x^2 - \frac{379}{5}x - 82$$

(3, 2)

(−1, 0.5)

(1, −1)

(2, −0.5)

[interactive]

Application

Best fit linear function

What least squares problem $Ax = b$ finds the best linear function $f(x, y)$ fitting the following data?

The general equation for a linear function in two variables is

$$f(x, y) = Ax + By + C.$$

x	y	$f(x, y)$
1	0	0
0	1	1
−1	0	3
0	−1	4

So we want to solve

$$\begin{aligned}
A(1) + \quad B(0) + C &= 0 \\
A(0) + \quad B(1) + C &= 1 \\
A(-1) + \quad B(0) + C &= 3 \\
A(0) + B(-1) + C &= 4
\end{aligned}$$

In matrix form:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}.$$

Answer:

$$f(x, y) = -\frac{3}{2}x - \frac{3}{2}y + 2$$

[interactive]

$f(x, y)$

$(0, -1, 4)(-1, 0)$

$f(0, -1)(-1, 0, 3)$

$(0, 1, 1)$

$f(1, 0)$ $f(0, 1)$

$(1, 0, 0)$

Graph of

$$f(x, y) = -\frac{3}{2}x - \frac{3}{2}y + 2$$
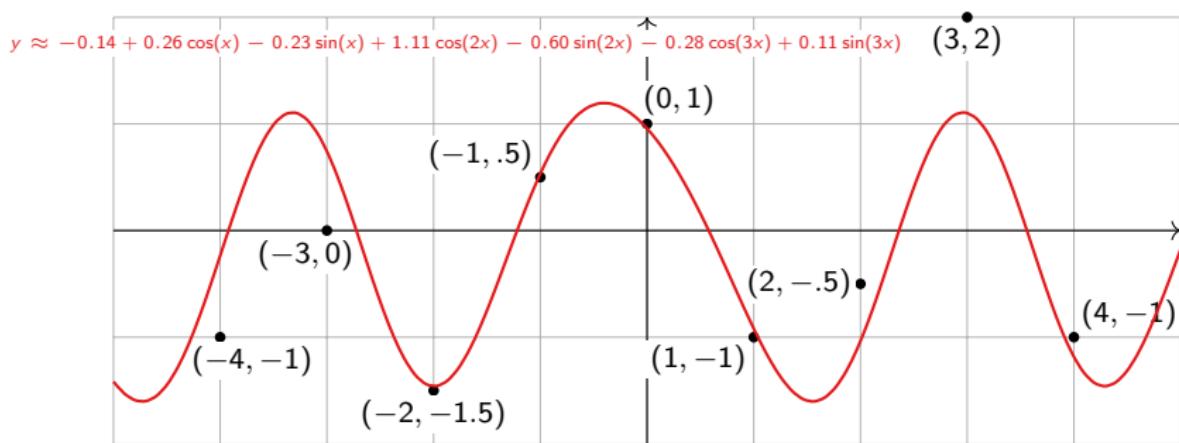
For fun: what is the best-fit function of the form

$$y = A + B\cos(x) + C\sin(x) + D\cos(2x) + E\sin(2x) + F\cos(3x) + G\sin(3x)$$

passing through the points

$$\begin{pmatrix} -4 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1.5 \end{pmatrix}, \begin{pmatrix} -1 \\ .5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -.5 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix}?$$

$y \approx -0.14 + 0.26\cos(x) - 0.23\sin(x) + 1.11\cos(2x) - 0.60\sin(2x) - 0.28\cos(3x) + 0.11\sin(3x)$

[interactive]

Summary

- A **least squares solution** of $Ax = b$ is a vector $\widehat{x}$ such that $\widehat{b} = A\widehat{x}$ is *as close to b as possible*.
- This means that $\widehat{b} = b_{\text{Col } A}$.
- One way to compute a least squares solution is by solving the system of equations
$$(A^T A)\widehat{x} = A^T b.$$
  Note that $A^T A$ is a (symmetric) square matrix.
- Least-squares solutions are unique when the columns of A are linearly independent.
- You can use least-squares to find best-fit lines, parabolas, ellipses, planes, etc.