Announcements Nov 1

- Masks ⇝ Thank you!
- WeBWorK 3.5 & 3.6 due Tue @ midnight
- ~~Studio but no~~ quiz Friday 3.5 & 3.6?
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Chapter 5

Eigenvectors and eigenvalues

Where are we?

Remember:

Almost every engineering problem, no matter how huge, can be reduced to linear algebra:

$$Ax = b \quad \text{or}$$

$$* \quad Ax = \lambda x \quad *$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), fluid mixing, RLC circuits, clustering (data analysis), principal component analysis, Google, Netflix (collaborative prediction), infectious disease models, special relativity, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

A Question from Biology

In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year - think of it as a vector (f, s, t) - what is the population the next year?

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix}$$

input $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \longrightarrow$ output $\begin{pmatrix} 28 \\ 1 \\ 1 \end{pmatrix}$

Now choose some starting population vector $u = (f, s, t)$ and choose some number of years N. What is the new population after N years?

Demo

Section 5.1

Eigenvectors and eigenvalues

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

e.g. $v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$, $\lambda = 2$ in last example

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

In simpler terms: Av is a scalar multiple of v.

In other words: Av points in the same direction as v.

Think of this in terms of inputs and outputs!

eigen $=$ *characteristic (or: self)*

This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

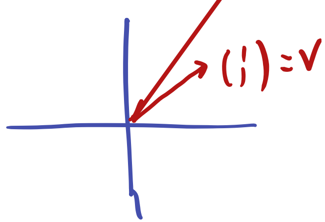
$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

Can you find any eigenvectors/eigenvalues for the following matrix?

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

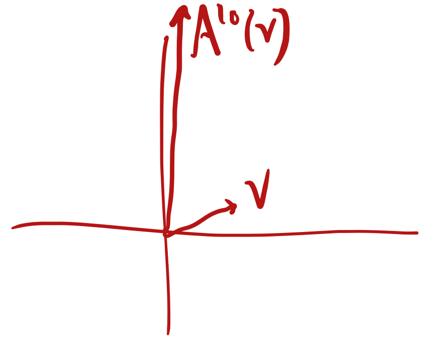
What happens when you apply larger and larger powers of A to a vector?



Rabbits

What's up with them?

Eigenvectors and Eigenvalues



When we apply large powers of the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

to a vector v not on the x-axis, we see that $A^n v$ gets closer and closer to the y-axis, and it's length gets approximately tripled each time. This is because the largest eigenvalue is 3 and its eigenspace is the y-axis.

For the rabbit matrix

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

We will see that 2 is the largest eigenvalue, and its ~~eigenspace~~ vectors with that eigenvalue. is the span of the vector $(16, 4, 1)$. That's why all populations of rabbits tend towards the ratio 16:4:1 and why the population approximately doubles each year.

Eigenvectors and Eigenvalues

Examples

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}, \quad \lambda = 2$$

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

↑ not eigenvector!

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4$$

$$\begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

How do you check?

Poll

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?

yes! $\lambda = 2$

yes! $\lambda = 0$ Null space.

no!

no! 0 never is an eigenvector

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Is $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ a multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

$\begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} \begin{pmatrix} 200 \\ 100 \end{pmatrix}$

Eigenvectors and Eigenvalues

Confirming eigenvalues

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

(Any eigenvector you find is called a 3-eigenvector.)

Never need to do again.

$$Av = 3v \rightsquigarrow Av = 3Iv \rightsquigarrow Av - 3Iv = 0 \rightsquigarrow (A - 3I)v = 0.$$

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

$Ax = 0$ problem with $A - 3I$ instead of A.

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & -4 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} \rightsquigarrow x + 4y = 0 \rightsquigarrow \begin{aligned} x &= -4y \\ y &= y \end{aligned}$$

$$\rightsquigarrow \begin{pmatrix} -4 \\ 1 \end{pmatrix} \text{ or any multiple.}$$

Check: $\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix}$

What is a general procedure for finding eigenvalues?

Eigenvectors and Eigenvalues
Confirming eigenvalues

The following are equivalent:
- λ is an eigenvalue of A
- $\mathrm{Nul}(A - \lambda I)$ is nontrivial $\longleftrightarrow \det A - \lambda I = 0$.

So the recipe for checking if λ is an eigenvalue of A is:
- subtract λ from the diagonal entries of A
- row reduce
- check if there are fewer than n pivots

Confirm that $\lambda = 1$ is not an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ -1 & -2 \end{pmatrix}$$

2 pivots

$\rightsquigarrow \mathrm{Nul}(A - 1 \cdot I) = 0$

$\rightsquigarrow$ 1 not eigenvalue.

Eigenspaces

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

Why is this a subspace?

Fact. λ-eigenspace for $A = \mathrm{Nul}(A - \lambda I)$

Example. Find the eigenspaces for $\lambda = 2$ and $\lambda = -1$ and sketch.

$$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

2-eigenspace

$$\begin{pmatrix} 3 & -6 \\ 3 & -6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

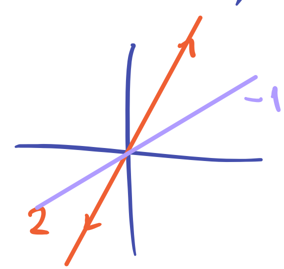
2x2 trick $\rightsquigarrow$ Span $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

swap entries in top row, negate one.

$$\begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

-1-eigenspace

$$\begin{pmatrix} 6 & -6 \\ 3 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$



Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

Eigenspaces

Bases

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenspaces

Bases

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Eigenspaces

Bases

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$

Eigenvalues

And invertibility

Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A

Why?

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Important! You can not find the eigenvalues by row reducing first! After you find the eigenvalues, you row reduce $A - \lambda I$ to find the eigenspaces. But once you start row reducing the original matrix, you change the eigenvalues.

Fact. If $v_1 \ldots v_k$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots \lambda_k$, then $\{v_1, \ldots, v_k\}$ are linearly independent.

Why?

Consequence. An $n \times n$ matrix has at most n distinct eigenvalues.

Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

- Reflection about a line in $\mathbb{R}^2$ (doesn't matter which line!)
- Orthogonal projection onto a line in $\mathbb{R}^2$ (doesn't matter which line!)
- Scaling of $\mathbb{R}^2$ by 3
- (Standard) shear of $\mathbb{R}^2$
- Orthogonal projection to a plane in $\mathbb{R}^3$ (doesn't matter which plane!)

▸ Demo

Eigenvalues for rotations?

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)?

Demo

Summary of Section 5.1

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- Recipe: The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A
- Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations

Typical exam questions 5.1

- Find the 2-eigenvectors for the matrix

$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

- True or false: The zero vector is an eigenvector for every matrix.
- How many different eigenvalues can there be for an $n \times n$ matrix?
- Consider the reflection of $\mathbb{R}^2$ about the line $y = 7x$. What are the eigenvalues (of the standard matrix)?
- Consider the $\pi/2$ rotation of $\mathbb{R}^3$ about the z-axis. What are the eigenvalues (of the standard matrix)?

Section 5.2

The characteristic polynomial

Characteristic polynomial

Recall:

$$\lambda \text{ is an eigenvalue of } A \iff A - \lambda I \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial, the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

The eigenrecipe

Say you are given a square matrix A.

Step 1. Find the eigenvalues of A by solving

$$\det(A - \lambda I) = 0$$

Step 2. For each eigenvalue λ_i the λ_i-eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

To find a basis, find the vector parametric solution, as usual.

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$