

# Eigenspaces

## Bases

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

## Announcements Nov 3

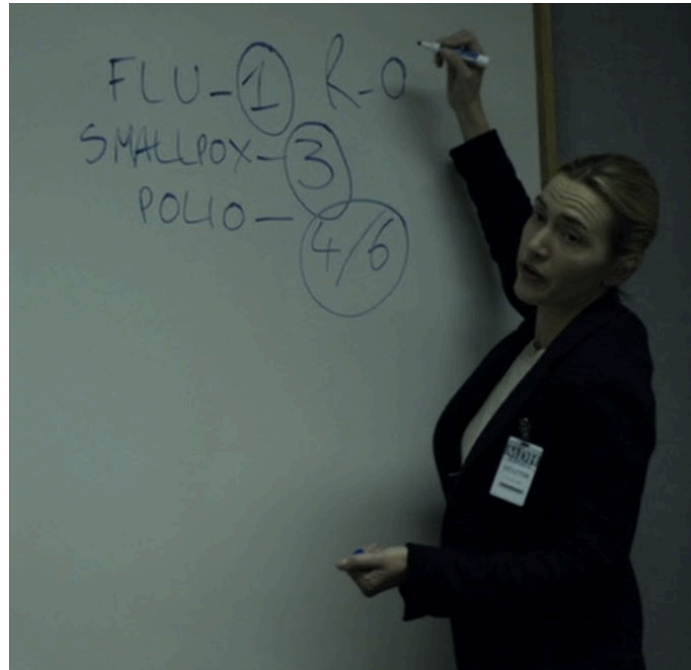
- Masks  $\rightsquigarrow$  Thank you!
  - WeBWorK 5.1 & 5.2 due **Tue @ midnight**
  - Studio and **Quiz Friday** on 3.5, 3.6, 4.1, 4.2, 4.3
  - Office hrs: Tue 4-5 Teams + **Thu 1-2 Skiles courtyard/Teams** + Pop-ups
  - **Midterm 3 Nov 17** 8–9:15 on Teams, Sec. 3.5–5.5
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- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab <https://tutoring.gatech.edu/drop-in/>
- Section M web site: Google “Dan Margalit math”, click on 1553
  - ▶ future blank slides, past lecture slides, advice
- Old exams: Google “Dan Margalit math”, click on Teaching
- Tutoring: <http://tutoring.gatech.edu/tutoring>
- Counseling center: <https://counseling.gatech.edu>
- Use Piazza for general questions
- You can do it!

$R_0$

# $R_0$

For a given virus,  $R_0$  is the average number of people that each infected person infects. If  $R_0$  is large, that is bad. Patient zero infects  $R_0$  people, who then infect  $R_0^2$  people, who then infect  $R_0^3$  people. That is exponential growth. (If  $R_0$  is less than 1, then that's good.)



# $R_0$

For a given virus,  $R_0$  is the average number of people that each infected person infects. If  $R_0$  is large, that is bad. Patient zero infects  $R_0$  people, who then infect  $R_0^2$  people, who then infect  $R_0^3$  people. That is exponential growth.

Whenever we see an exponential growth rate, we should think: eigenvalue.

It turns out that  $R_0$  is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment. That's a matrix. The largest eigenvalue is  $R_0$ .

## $R_0$ is an eigenvalue

It turns out that  $R_0$  is an eigenvalue. The rough idea is very similar to our rabbit example: split the population into compartments, figure out how often each compartment infects each other compartment.

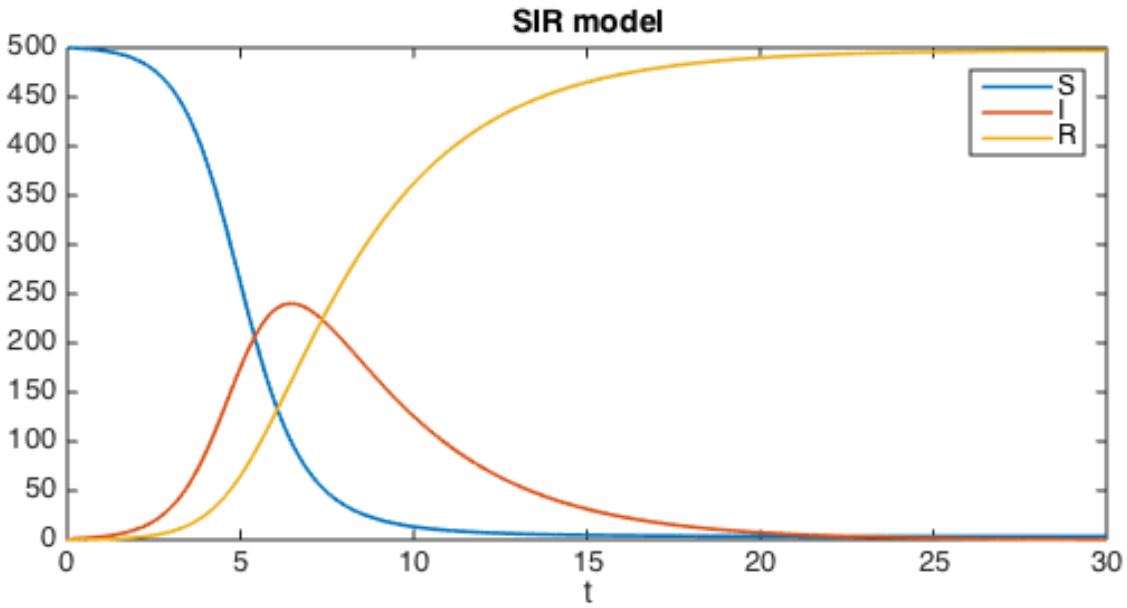
For malaria, the compartments might be mosquitoes and humans.

For a sexually transmitted disease in a heterosexual population, the compartments might be males and females.



# Bell curves

The growth rate of infection does not stay exponential forever, because the recovered population has immunity. That's where you get these bell curves.





# Section 5.1

## Eigenvectors and eigenvalues

# Eigenvectors and Eigenvalues

Suppose  $A$  is an  $n \times n$  matrix and there is a  $v \neq 0$  in  $\mathbb{R}^n$  and  $\lambda$  in  $\mathbb{R}$  so that

$$Av = \lambda v$$

then  $v$  is called an **eigenvector** for  $A$ , and  $\lambda$  is the corresponding **eigenvalue**.

In simpler terms:  $Av$  is a scalar multiple of  $v$ .

In other words:  $Av$  points in the same direction as  $v$ .

Think of this in terms of inputs and outputs!

*eigen = characteristic (or: self)*

This the the most important definition in the course.

▶ Demo

# Eigenspaces

Bases

Find a basis for the 2-eigenspace:

all eigenvec's  
with eigenval 2  
plus 0 vector.

$$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

$Av = 2v$  same as...  $v$  in  $\text{Nul}(A - 2I)$

$$A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{pmatrix} \boxed{1} & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

subtract 2  
from diag.

$$\rightarrow \left\{ \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\text{Check: } \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

# Eigenspaces

## Bases

Find a basis for the 2-eigenspace:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

← scales by 2.

$$A - 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$x = x$$

$$y = y$$

$$x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2-eigenspace:  $\mathbb{R}^2$

# Eigenspaces

## Bases

Find a basis for the 2-eigenspace:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

2-eigensp.  
is  $xy$ -plane.

# Eigenspaces

## Bases

Find a basis for the 2-eigenspace:

$$\begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} x=x \\ y=0 \end{matrix} \rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \checkmark$$

# Eigenvalues

And invertibility

Fact.  $A$  invertible  $\Leftrightarrow 0$  is not an eigenvalue of  $A$

Why?

$0$  is an eigenvalue of  $A$

$$\Leftrightarrow Av = 0 \cdot v = 0 \quad \begin{array}{l} \text{Some} \\ v \neq 0 \end{array}$$

$$\Leftrightarrow \text{Nul}(A) \neq \{0\} \stackrel{\text{IVT}}{\Leftrightarrow} A \text{ not invertible.}$$

# Eigenvalues

Triangular matrices

$$\lambda \text{ eigenval} \iff \text{Nul}(A - \lambda I) \neq 0 \iff A - \lambda I \text{ fewer than } n \text{ pivots.}$$

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

$$\begin{pmatrix} 5 & 9 \\ 0 & 7 \end{pmatrix}$$

0's below diagonal

Can see 7 is an eigenval. because

$$\begin{pmatrix} -2 & 9 \\ 0 & 0 \end{pmatrix} \text{ fewer than } n \text{ pivots.}$$

Same for 5

**Important!** You can not find the eigenvalues by row reducing first! After you find the eigenvalues, you row reduce  $A - \lambda I$  to find the eigenspaces. But once you start row reducing the original matrix, you change the eigenvalues.

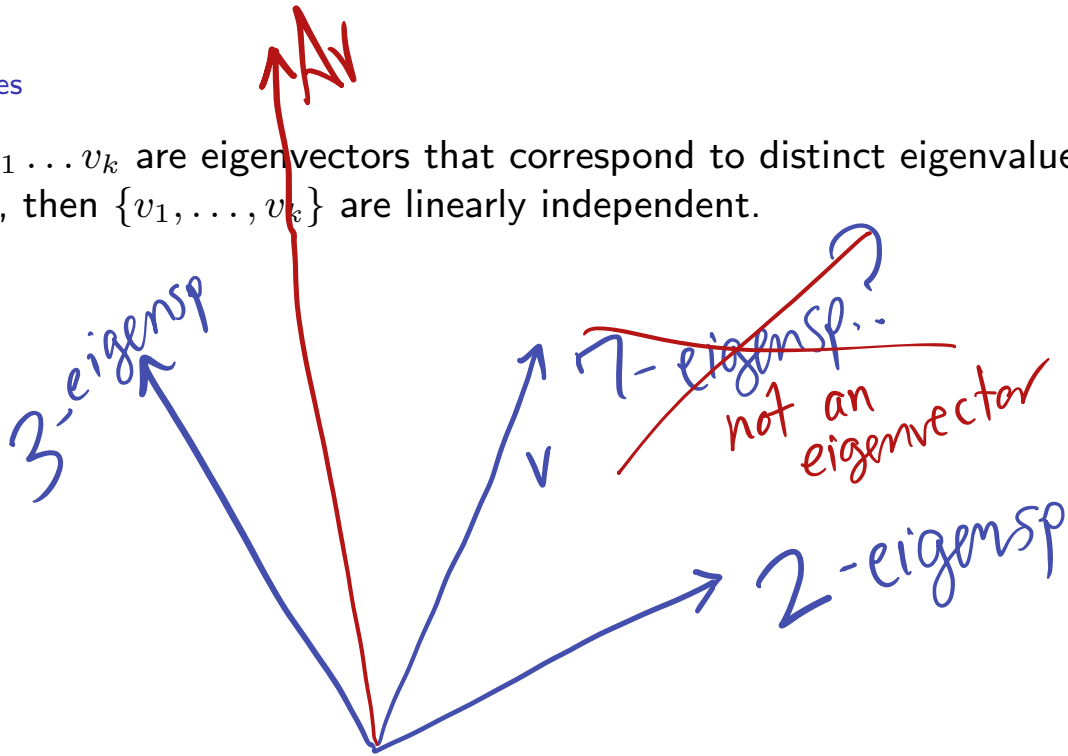


# Eigenvalues

## Distinct eigenvalues

**Fact.** If  $v_1 \dots v_k$  are eigenvectors that correspond to distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ , then  $\{v_1, \dots, v_k\}$  are linearly independent.

Why?



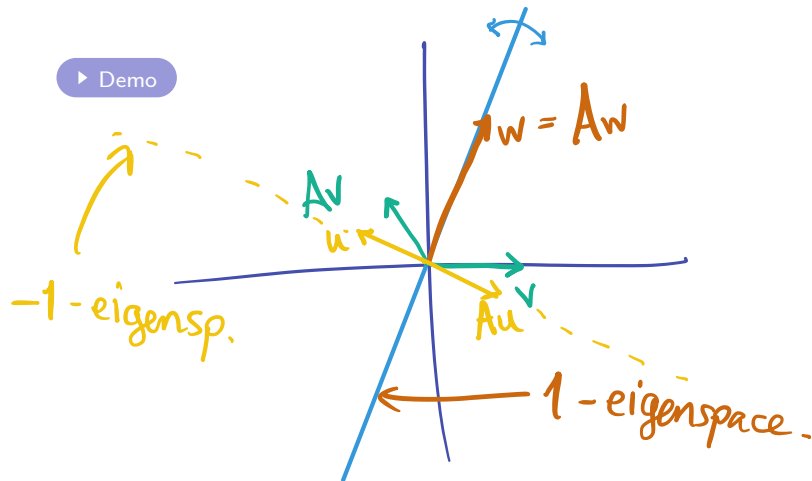
**Consequence.** An  $n \times n$  matrix has at most  $n$  distinct eigenvalues.

## Eigenvalues geometrically

If  $v$  is an eigenvector of  $A$  then that means  $v$  and  $Av$  are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

- Reflection about a line in  $\mathbb{R}^2$  (doesn't matter which line!)
- Orthogonal projection onto a line in  $\mathbb{R}^2$  (doesn't matter which line!)
- Scaling of  $\mathbb{R}^2$  by 3
- (Standard) shear of  $\mathbb{R}^2$
- Orthogonal projection to a plane in  $\mathbb{R}^3$  (doesn't matter which plane!)



## Eigenvalues for rotations?

If  $v$  is an eigenvector of  $A$  then that means  $v$  and  $Av$  are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of  $\mathbb{R}^2$  by  $\pi/2$  (counterclockwise)?

▶ Demo

# Section 5.2

## The characteristic polynomial

# Outline of Section 5.2

- How to find the eigenvalues, via the characteristic polynomial
- Techniques for the  $3 \times 3$  case

# Characteristic polynomial

*Recall:*

$\lambda$  is an eigenvalue of  $A \iff A - \lambda I$  is not invertible

So to find eigenvalues of  $A$  we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial, the **characteristic polynomial** of  $A$ .

The roots of the characteristic polynomial are the eigenvalues of  $A$ .

# The eigenrecipe

Say you are given a square matrix  $A$ .

Step 1. Find the eigenvalues of  $A$  by solving

$$\det(A - \lambda I) = 0$$

) New

Step 2. For each eigenvalue  $\lambda_i$  the  $\lambda_i$ -eigenspace is the solution to

$$(A - \lambda_i I)x = 0$$

) Sec 5.1

To find a basis, find the vector parametric solution, as usual.

# Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - 4$$
$$= \lambda^2 - 6\lambda + 1$$

$$\lambda = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}$$



# Two shortcuts for $2 \times 2$ eigenvectors

Find the eigenspaces for the eigenvalues on the last page. Two tricks.

(1) We do not need to row reduce  $A - \lambda I$  by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \lambda = 3 \pm 2\sqrt{2}$$

$$\lambda = 3 + 2\sqrt{2}$$

$$A - \lambda I = \begin{pmatrix} 2 - 2\sqrt{2} & 2 \\ 2 & -2 - 2\sqrt{2} \end{pmatrix} \xrightarrow{\textcircled{1}} \begin{pmatrix} 2 - 2\sqrt{2} & 2 \\ 0 & 0 \end{pmatrix} \xrightarrow{\textcircled{2}} \begin{pmatrix} -2 \\ 2 - \sqrt{2} \end{pmatrix}$$

$(3 + 2\sqrt{2})$ -  
eigenvec.  
 $\downarrow$

# Characteristic polynomial

Find the characteristic polynomial and eigenvalues of the Fibonacci matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0 \rightsquigarrow \lambda^2 - \lambda - 1 = 0$$

$$\rightsquigarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$$

golden ratio  $\varphi$   
&  $-\frac{1}{\varphi}$

Find eigenvectors  
using 2 tricks from last slide.

## 3 × 3 matrices

The 3 × 3 case is harder. There is a version of the quadratic formula for cubic polynomials, called Cardano's formula. But it is more complicated. It looks something like this:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} + \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}.$$

There is an even more complicated formula for quartic polynomials.

One of the most celebrated theorems in math, the Abel–Ruffini theorem, says that there is no such formula for quintic polynomials.

# Characteristic polynomials

$3 \times 3$  matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ -3 & 0 & -1 \end{pmatrix}$$

What are the eigenvalues? Hint: Don't multiply everything out!

$$\det \begin{pmatrix} 7-\lambda & 0 & 3 \\ -3 & 2-\lambda & -3 \\ -3 & 0 & -1-\lambda \end{pmatrix} =$$

$$(7-\lambda)((2-\lambda)(-1-\lambda)) + 3(3(2-\lambda))$$

$$= (2-\lambda)(\text{quadratic}) \rightsquigarrow \lambda = 2$$

two more eigenvals

# Characteristic polynomials

$3 \times 3$  matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer:  $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

# Characteristic polynomials

$3 \times 3$  matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

*Hint:* We already know one eigenvalue! Polynomial long division  $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

# Characteristic polynomials

$3 \times 3$  matrices

Find the characteristic polynomial and eigenvalues.

$$\begin{pmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{pmatrix}$$

Characteristic polynomial:  $-\lambda^3 + 9\lambda^2 - 23\lambda + 15$

This time we don't know any of the roots! We can use the rational root theorem: any integer root of a polynomial with leading coefficient  $\pm 1$  divides the constant term.

So we plug in  $\pm 1$ ,  $\pm 3$ ,  $\pm 5$ ,  $\pm 15$  into the polynomial and hope for the best. Luckily we find that 1, 3, and 5 are all roots, so we found all the eigenvalues!

If we were less lucky and found only one eigenvalue, we could again use long division like on the last slide.

# Eigenvalues

## Triangular matrices

**Fact.** The eigenvalues of a triangular matrix are the diagonal entries.

*Why?*

**Warning!** You cannot find eigenvalues by row reducing and then using this fact. You need to work with the original matrix. Finding eigenspaces involves row reducing  $A - \lambda I$ , but there is no row reduction in finding eigenvalues.



# Characteristic polynomials, trace, and determinant

The **trace** of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

$$(-1)^n \lambda^n + (-1)^{n-1} \boxed{\text{trace}(A)} \lambda^{n-1} + \boxed{???} \lambda^{n-2} + \dots \boxed{???} \lambda + \boxed{\det(A)}$$

So for a  $2 \times 2$  matrix:

$$\lambda^2 - \text{trace}(A)\lambda + \det(A)$$

And for a  $3 \times 3$  matrix:

$$-\lambda^3 + \text{trace}(A)\lambda^2 - \boxed{???} \lambda + \det(A)$$

# Characteristic polynomials, trace, and determinant

The **trace** of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

$$(-1)^n \lambda^n + (-1)^{n-1} \boxed{\text{trace}(A)} \lambda^{n-1} + \boxed{???} \lambda^{n-2} + \dots \boxed{???} \lambda + \boxed{\det(A)}$$

**Consequence 1.** The constant term is zero  $\Leftrightarrow A$  is not invertible

**Consequence 2.** The determinant is the product of the eigenvalues.

# Algebraic multiplicity

The **algebraic multiplicity** of an eigenvalue  $\lambda$  is its multiplicity as a root of the characteristic polynomial.

*Example.* Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

**Fact.** The sum of the algebraic multiplicities of the (real) eigenvalues of an  $n \times n$  matrix is at most  $n$ .

## Summary of Section 5.2

- The characteristic polynomial of  $A$  is  $\det(A - \lambda I)$
- The roots of the characteristic polynomial for  $A$  are the eigenvalues
- Techniques for  $3 \times 3$  matrices:
  - ▶ Don't multiply out if there is a common factor
  - ▶ If there is no constant term then factor out  $\lambda$
  - ▶ If the matrix is triangular, the eigenvalues are the diagonal entries
  - ▶ Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - ▶ Use the geometry to determine an eigenvalue
- Given an square matrix  $A$ :
  - ▶ The eigenvalues are the solutions to  $\det(A - \lambda I) = 0$
  - ▶ Each  $\lambda_i$ -eigenspace is the solution to  $(A - \lambda_i I)x = 0$

## Typical Exam Questions 5.2

- True or false: Every  $n \times n$  matrix has an eigenvalue.
- True or false: Every  $n \times n$  matrix has  $n$  distinct eigenvalues.
- True or false: The nullity of  $A - \lambda I$  is the dimension of the  $\lambda$ -eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the  $n \times n$  zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

*Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues, and then factor out a linear term.*