Announcements Nov 8

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 5.1 & 5.2 due Tue @ midnight
- Studio and (Last) Quiz Friday on 5.1 & 5.2
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Eigenvalues in Structural Engineering

Watch this video about the Tacoma Narrows bridge. ▸ Watch

Here are some toy models. ▸ Check it out

The masses move the most at their natural frequencies ω. To find those, use the spring equation: $mx'' = -kx \quad \rightsquigarrow \quad \sin(\omega t)$.

With 3 springs and 2 equal masses, we get:

$$\begin{aligned} mx_1'' &= -kx_1 + k(x_2 - x_1) \\ mx_2'' &= -kx_2 + k(x_1 - x_2) \end{aligned}$$

Guess a solution $x_1(t) = A_1(\cos(\omega t) + i\sin(\omega t))$ and similar for x_2. Finding ω reduces to finding eigenvalues of $\begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix}$.

Eigenvectors: $(1,1)$ & $(1,-1)$ (in/out of phase) ▸ Details

Section 5.2

The characteristic polynomial

Algebraic multiplicity

$f(x) = (x-1)^2$ roots: 1 mult 2

$g(x) = (x+1)(x-1)$ roots: ± 1 mult 1

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

$Ae_1 = 5e_1$

$Ae_4 = 5e_4$

$$(5-\lambda)(-\lambda)(-1-\lambda)(5-\lambda)$$

$$= (5-\lambda)^2(-\lambda)(-1-\lambda)$$

5 has mult. 2.

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

Section 5.4

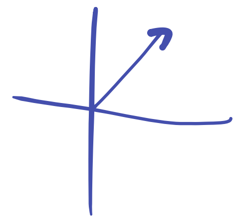
Diagonalization

We understand diagonal matrices

We completely understand what diagonal matrices do to $\mathbb{R}^n$. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

scales x-dir by 2
y-dir by 3



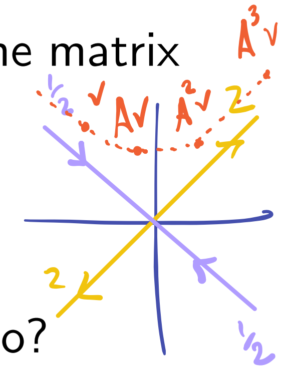
We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If A is diagonal, powers of A are easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 3^{10} \end{pmatrix}$$

Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose want to understand the matrix



$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

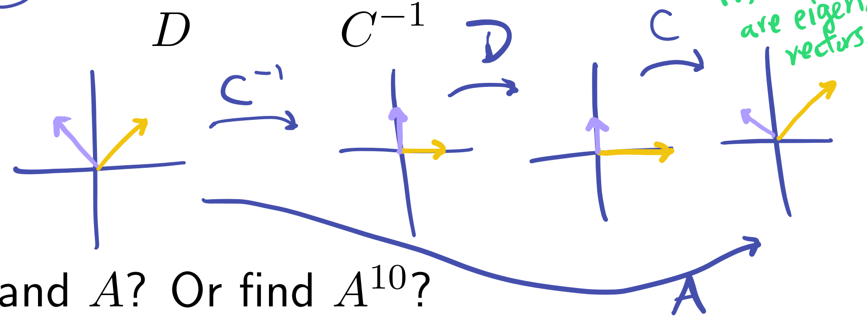
geometrically? Or take it's 10th power? What would we do?

What if I give you the following equality:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A \quad = \quad C \quad D \quad C^{-1}$$

This is called diagonalization.



How does this help us understand A? Or find A^{10}?

Powers of matrices that are similar to diagonal ones

What if I give you the following equality:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A \quad = \quad C \quad D \quad C^{-1}$$

This is called diagonalization.

How does this help us understand A? Or find A^{10}? Demo

$$A^2 = (CDC^{-1})(CDC^{-1}) = CD^2C^{-1}$$

$$A^{10} = CD^{10}C^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{10} & 0 \\ 0 & (1/2)^{10} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if we can write:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

We say that A is similar to D. Sec 5.3

How does this factorization of A help describe what A does to $\mathbb{R}^n$?
How does this help us take powers of A?

Understanding the rabbit example: since 2 is the largest eigenvalue, (almost) all other vectors get pulled towards that eigenvector. Compare with the example from the last slide.

Diagonalization

The recipe

$A = CDC^{-1}$ where D is diagonal,

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

$$= \qquad C \qquad\qquad D \qquad\qquad C^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, with multiplicity, in order.

Why?

Example

Diagonalize if possible.

$$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

eigenvals: 2, −1 ⇝ eigenvecs must be indep.

2-eigenvecs: $A-2I = \begin{pmatrix} 0 & 6 \\ 0 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(−1)-eigenvecs: $A-(-I) = \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$A = \underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1}}_{C^{-1}}$$

Variations:
① flip order of cols in C & entries of D
② scale cols of C by nonzero scalar

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

eigenvals: 3 (mult 2)

3-eigenvecs: $A-3I=\begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 1D eigenspace.

Not diagonalizable: don't have 2 indep eigenvectors.

Example

Diagonalize if possible.

$$A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

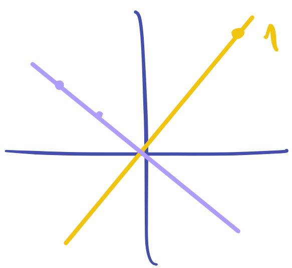
$\det(A - \lambda I) =$ quad. poly in λ

▸ Demo

Hint: the eigenvalues are 1 and $1/2$

1 eigensp: $\begin{pmatrix} -1/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1/2 eigensp: $\begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Fibonacci numbers

for fun.

Diagonalize the matrix.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues are φ *&* $-1/\varphi$*, with eigenvectors* $(\varphi, 1)$ *&* $(-1/\varphi, 1)$

What does this tell us about Fibonacci numbers? How quickly do they grow? What is the ratio between consecutive Fibonacci numbers?

Use this to give a formula for the nth Fibonacci number

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Hint: the eigenvalues (with multiplicity) are $3, -1, 1$ and $2, 2, 1$

Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why? If eigenvals are distinct, the n eigenvecs must be lin ind.

Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $1 \leq d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

So the recipe for checking diagonalizability is:

- If there are not n eigenvalues with multiplicity, then stop.
- For each eigenvalue with alg. mult. greater than 1, check if the geometric multiplicity is equal to the algebraic multiplicity. If any of them are smaller, the matrix is not diagonalizable.
- Otherwise, the matrix is diagonalizable.

More rabbits

Which ones are diagonalizable?

$$\begin{pmatrix} 0 & 4 \\ \frac{1}{2} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 4 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hint: the characteristic polynomials are $-\lambda^3 + 3\lambda + 2$ *and* $-\lambda^3 + 2\lambda + 1$ *and both have rational roots.*

Interpret all of these as rabbit matrices. What can you say about the rabbit populations?

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Section 5.2

The characteristic polynomial

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the following matrix.

$$\begin{pmatrix} 7 & 0 & 3 \\ -3 & 2 & -3 \\ 4 & 2 & 0 \end{pmatrix}$$

Answer: $-\lambda^3 + 9\lambda^2 - 8\lambda$

What are the eigenvalues?

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Answer:

$$-\lambda^3 + 3\lambda + 2$$

What are the eigenvalues?

Hint: We already know one eigenvalue! Polynomial long division $\rightsquigarrow$

$$(\lambda - 2)(-\lambda^2 - 2\lambda - 1)$$

Don't really need long division: the first and last terms of the quadratic are easy to find; can guess and check the other term.

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial and eigenvalues.

$$\begin{pmatrix} 5 & -2 & 2 \\ 4 & -3 & 4 \\ 4 & -6 & 7 \end{pmatrix}$$

Characteristic polynomial: $-\lambda^3 + 9\lambda^2 - 23\lambda + 15$

This time we don't know any of the roots! We can use the rational root theorem: any integer root of a polynomial with leading coefficient ± 1 divides the constant term.

So we plug in ± 1, ± 3, ± 5, ± 15 into the polynomial and hope for the best. Luckily we find that 1, 3, and 5 are all roots, so we found all the eigenvalues!

If we were less lucky and found only one eigenvalue, we could again use long division like on the last slide.

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

$$(-1)^n\lambda^n + (-1)^{n-1}\boxed{\text{trace(A)}}\lambda^{n-1} + \boxed{???}\lambda^{n-2} + \cdots \boxed{???}\lambda + \boxed{\text{det(A)}}$$

So for a 2×2 matrix:

$$\lambda^2 - \mathrm{trace}(A)\lambda + \det(A)$$

And for a 3×3 matrix:

$$-\lambda^3 + \mathrm{trace}(A)\lambda^2 - \boxed{???}\lambda + \det(A)$$

Characteristic polynomials, trace, and determinant

The trace of a matrix is the sum of the diagonal entries.

The characteristic polynomial always looks like:

$$(-1)^n\lambda^n + (-1)^{n-1}\boxed{\text{trace(A)}}\lambda^{n-1} + \boxed{???}\lambda^{n-2} + \cdots \boxed{???}\lambda + \boxed{\det(A)}$$

Consequence 1. The constant term is zero $\Leftrightarrow$ A is not invertible

Consequence 2. The determinant is the product of the eigenvalues.

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

Fact. The sum of the algebraic multiplicities of the (real) eigenvalues of an $n \times n$ matrix is at most n.

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Typical Exam Questions 5.2

- True or false: Every $n \times n$ matrix has an eigenvalue.
- True or false: Every $n \times n$ matrix has n distinct eigenvalues.
- True or false: The nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the $n \times n$ zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues, and then factor out a linear term.