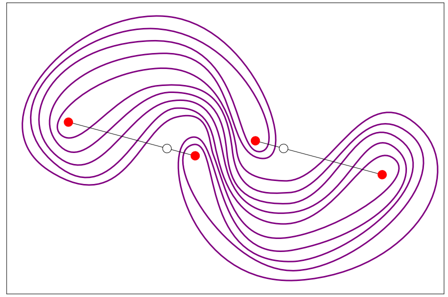
Announcements Nov 10

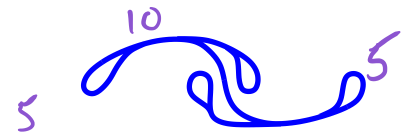
- Masks $\rightsquigarrow$ Thank you!
- Studio and (Last) Quiz Friday on 5.1 & 5.2
- WeBWorK 5.4 & 5.5 due Tue @ midnight
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Review sessions: Prof. M Mon and Wed 4:30–5:15 Howey L1
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Taffy pullers

How efficient is this taffy puller?





If you run the taffy puller, the taffy starts to look like the shape on the right. Every rotation of the machine changes the number of strands of taffy by a matrix:

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

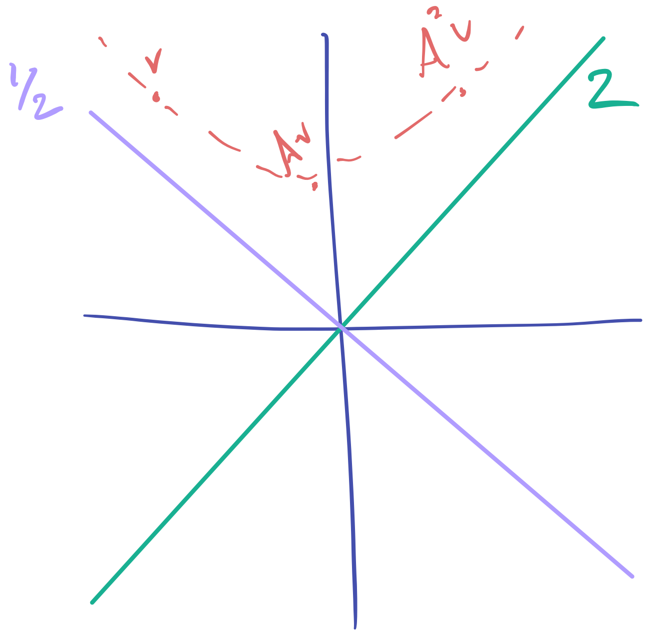
The largest eigenvalue λ of this matrix describes the efficiency of the taffy puller. With every rotation, the number of strands multiplies by λ.

Section 5.4

Diagonalization

We understand matrices when we know their eigendata

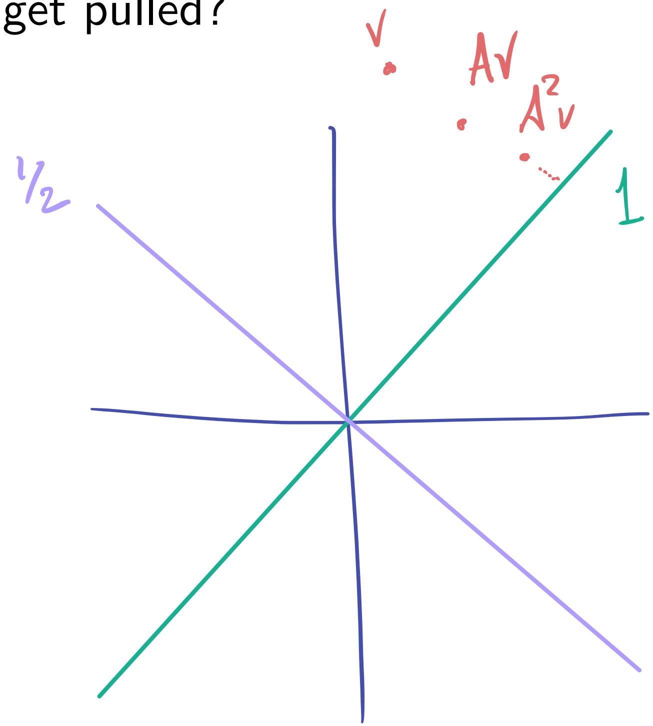
Suppose that A is a 2×2 matrix with eigenvalues 2 and $1/2$, with 2-eigenvector $(1, 1)$ and $1/2$-eigenvector $(-1, 1)$. What does A do to $\mathbb{R}^2$? Choose a vector v and find $Av, A^2v, \ldots$ In which direction do most vectors get pulled?



Demo

We understand matrices when we know their eigendata

Suppose that A is a 2×2 matrix with eigenvalues 1 and $1/2$, with 1-eigenvector $(1, 1)$ and $1/2$-eigenvector $(-1, 1)$. What does A do to $\mathbb{R}^2$? Choose a vector v and find $Av, A^2v, \ldots$ In which direction do most vectors get pulled?



Demo

We understand matrices when we know their eigendata

The moral of the last two examples is that if we have an $n \times n$ matrix, and if it has

1. n (real) eigenvalues and
2. n linearly independent eigenvectors

and if we know all of this data, then we have a clear picture of what the matrix does to $\mathbb{R}^n$.

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if we can write:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

We say that A is similar to D.

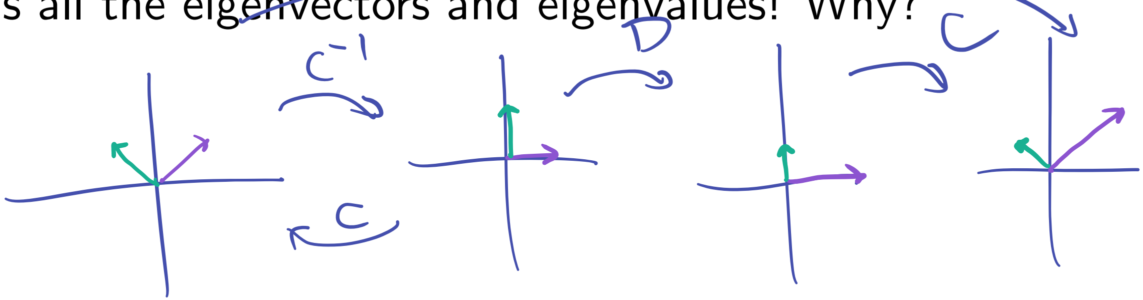
eigenvecs   eigenvals

Example:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A \qquad = \qquad C \qquad\qquad D \qquad\qquad C^{-1}$$

This tells us all the eigenvectors and eigenvalues! Why?



Demo

Powers of matrices that are similar to diagonal ones

Again, given the equality:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A \quad = \quad C \quad D \quad C^{-1}$$

How can we find A^{100} without doing 99 matrix calculations?

$$A^{100} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{100} & 0 \\ 0 & 1/2^{100} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Moral: Diagonalizable matrices behave a lot like diagonal matrices.

Diagonalization

The recipe

algebra geometry

Theorem. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

In this case

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}^{-1}$$

$$= \qquad C \qquad\qquad D \qquad\qquad C^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, with multiplicity, in order.

In other words, knowing the diagonalization is the same as knowing n eigenvalues and n independent eigenvectors.

More Examples

Diagonalize if possible.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad \cancel{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}}$$

Hint: the eigenvalues (with multiplicity) are $3, -1, 1$ and $2, 2, 1$

$$A = C \quad D \quad C^{-1}$$

1-eigenvec

$$\Big(\ \square\ \square\ \square\ \Big)\begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \text{same matrix as on left} \end{pmatrix}^{-1}$$

3-eigenvec

-1 eigenvec

To find 3-eigenvector: $\mathrm{Null}\begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \mathrm{Null}(A-3I)$

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

diag'able   not diag'able

Hint: the eigenvalues (with multiplicity) are $3, -1, 1$ and $2, 2, 1$

Maybe diag'able, maybe not.

All hinges on 2-eigensp: must be 2D

$\boxed{2D}$

2-eigensp: $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{matrix} x = -z \\ y = y \\ z = z \end{matrix} \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

So diag'able

1-eigenv.

$$A = \begin{pmatrix} 0 & -1 & \boxed{} \\ 1 & 0 & \\ 0 & 1 & \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & \boxed{\text{same}} \\ 1 & 0 & \\ 0 & 1 & \end{pmatrix}^{-1}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

Distinct eigenval's

$\rightsquigarrow$ lin ind eigenvec's

Non-Distinct Eigenvalues

Example from last time:

$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

eigenval: 2 mult 2

dim of eigensp $= 1 < 2$

$\Longrightarrow$ not diag'able.

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $1 \leq d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

Not diag'able:
$d_i < a_i$ some i.

So the recipe for checking diagonalizability is:

- If there are not n eigenvalues with multiplicity, then stop.
- For each eigenvalue with alg. mult. greater than 1, check if the geometric multiplicity is equal to the algebraic multiplicity. If any of them are smaller, the matrix is not diagonalizable.
- Otherwise, the matrix is diagonalizable.

More rabbits

You do!

Which ones are diagonalizable?

$$\begin{pmatrix} 0 & 4 \\ \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 4 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Hint: the characteristic polynomials are $-\lambda^3 + 3\lambda + 2$ *and* $-\lambda^3 + 2\lambda + 1$ *and both have rational roots.*

Interpret all of these as rabbit matrices. What can you say about the rabbit populations?

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\left(\begin{smallmatrix} 2 & 6 \\ 0 & -1 \end{smallmatrix}\right)$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

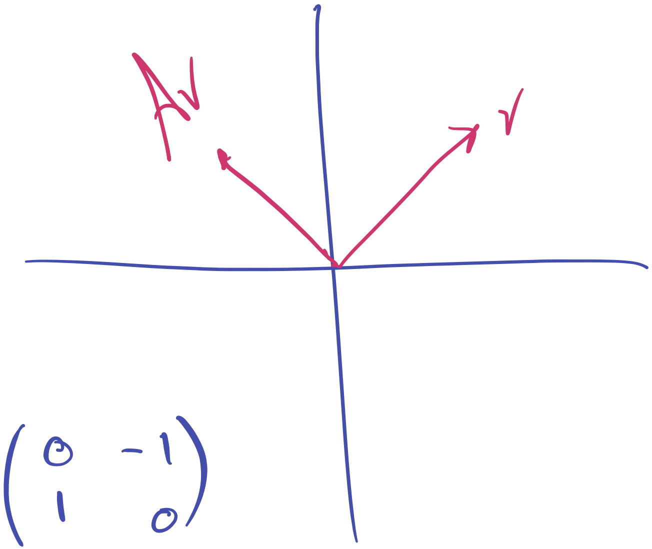
Section 5.1

Eigenvectors and eigenvalues

Eigenvalues for rotations?

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

What are the eigenvectors and eigenvalues for rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)?



none!

Demo

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + 1$$

No real solns!

Section 5.5

Complex Eigenvalues

Outline of Section 5.5

- Rotation matrices have no eigenvectors
- Crash course in complex numbers
- Finding complex eigenvectors and eigenvalues
- Complex eigenvalues correspond to rotations + dilations

▸ Demo ▸ Demo

A matrix without an eigenvector

Recall that rotation matrices like

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

have no eigenvectors. Why?

2 sec's ago

Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can't take the square root of a negative number:

$$x^2 + 1 = 0$$

$X^2 = -1$

$X = \pm\sqrt{-1} = \pm i$

Solution. Take square roots of negative numbers:

$$x = \pm\sqrt{-1}$$

We usually write $\sqrt{-1}$ as i (for "imaginary"), so $x = \pm i$.

Now try solving these:

$$x^2 + 3 = 0$$

$X = \pm\sqrt{-3} = \pm\sqrt{3}\sqrt{-1} = \pm\sqrt{3}\,i$

$$x^2 - x + 1 = 0$$

$\dfrac{+1 \pm \sqrt{1-4}}{2} = \dfrac{+1 \pm \sqrt{-3}}{2} = \dfrac{+1 \pm \sqrt{3}\,i}{2}$

Complex numbers

We can add/multiply (and divide!) complex numbers:

$(2-3i)+(-1+i) =$ $1-2i$

$(2-3i)(-1+i) =$ $-2+3i+2i-3i^2$

$-2+5i+3$

$1+5i$

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can conjugate complex numbers: $\overline{a + bi} = a - bi$

$$\overline{7 - 3i} = 7 + 3i$$

$$\overline{7 + 3i} = 7 - 3i$$

Complex numbers and polynomials

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots (counted with multiplicity).

Fact. If z is a root of a real polynomial then $\overline{z}$ is also a root.

So what are the possibilities for degree 2, 3 polynomials? deg 2 : 2 real roots or 2 complex

What does this have to do with eigenvalues of matrices? deg 3 : 1 or 3 real roots

$n \times n$ matrices

has n eigenvals (with multiplicity)

Complex eigenvalues

Say A is a square matrix with real entries. $A = \overline{A}$

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

Why?

$$Av = \lambda v$$

$$A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}$$

Trace and determinant

Now that we have complex eigenvalues, we have the following fact.

Fact. The sum of the eigenvalues of A (with multiplicity) is the trace of A and the product of the eigenvalues of A (with multiplicity) is the determinant.

Indeed, by the fundamental theorem of algebra, the characteristic polynomial factors as:

$$(x_1 - \lambda)(x_2 - \lambda) \cdots (x_n - \lambda).$$

From this we see that the product of the eigenvalues $x_1 x_2 \cdots x_n$ is the constant term, which we said was the determinant, and the sum $x_1 + x_2 + \cdots + x_n$ is $(-1)^{n-1}$ times the λ^{n-1} term, which we said was the trace.

Complex eigenvalues

$-i = \overline{i}$

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\lambda^2 + 1 = 0 \rightsquigarrow \lambda = \pm i$

i-eigenspace $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \overset{\text{trick \#1}}{\rightsquigarrow} \begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \underset{\#2}{\overset{\text{trick}}{\rightsquigarrow}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

trick 3

$-i$-eigenspace $\overline{\begin{pmatrix} 1 \\ -i \end{pmatrix}} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Three shortcuts for complex eigenvectors

Suppose we have a 2×2 matrix with complex eigenvalue λ.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.

Complex eigenvalues

Find the complex eigenvalues and eigenvectors for

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

you!

Summary of Section 5.5

- Complex numbers allow us to solve all polynomials completely, and find n eigenvalues for an $n \times n$ matrix, counting multiplicity
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$

Typical Exam Questions 5.5

- True/False. If v is an eigenvector for A with complex entries then $i \cdot v$ is also an eigenvector for A.
- True/False. If $(i, 1)$ is an eigenvector for A then $(i, -1)$ is also an eigenvector for A.
- If A is a 4×4 matrix with real entries, what are the possibilities for the number of non-real eigenvalues of A?
- Find the eigenvalues and eigenvectors for the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$