Announcements Nov 17

- Masks $\rightsquigarrow$ Thank you!
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5
- WeBWorK 5.6 & 6.1 due Tue @ midnight
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups

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- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Midterm 3 Review

Summary of Section 3.5

- A is invertible if there is a matrix B (called the inverse) with

$$AB = BA = I_n$$

- For a 2×2 matrix A we have that A is invertible exactly when $\det(A) \neq 0$ and in this case

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- If A is invertible, then $Ax = b$ has exactly one solution:

- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A \mid I_n)$.
- Invertible linear transformations correspond to invertible matrices.

Find the inverse of the matrix $\begin{pmatrix} 3 & 3 \\ 2 & 1 \end{pmatrix}$

Typical Exam Questions 3.5

- Find the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

- Find a 2×2 matrix with no zeros that is equal to its own inverse.
- Solve for the matrix X. Assume that all matrices that arise are invertible:

$$C + BX = A$$

- True/False. If A is invertible, then A^2 is invertible?
- Which linear transformation is the inverse of the clockwise rotation of $\mathbb{R}^2$ by $\pi/4$?
- True/False. The inverse of an invertible linear transformation must be invertible.
- Find a matrix with no zeros that is not invertible.
- Are there two different rabbit populations that will lead to the same population in the following year?

Summary of Section 3.6

- Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.
  (1) A is invertible
  (2) T is invertible
  (3) The reduced row echelon form of A is I_n
  (4) etc.

Typical Exam Questions Section 3.6

In all questions, suppose that A is an $n \times n$ matrix and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. For each question, answer YES or NO.

(1) Suppose that the reduced row echelon form of A does not have any zero rows. Must it be true that $Ax = b$ is consistent for all b in $\mathbb{R}^n$?

(2) Suppose that T is one-to-one. Is is possible that the columns of A add up to zero?

(3) Suppose that $Ax = e_1$ is not consistent. Is it possible that T is onto?

(4) Suppose that $n = 3$ and that $T\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 0$. Is it possible that T has exactly two pivots?

(5) Suppose that $n = 3$ and that T is one-to-one. Is it possible that the range of T is a plane?

Summary of Sections 4.1 and 4.3

Say det is a function det : {matrices} $\to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function det and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\dfrac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S)$.

Typical Exam Questions 4.1 and 4.3

- Find the value of h that makes the determinant 0:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & h \end{pmatrix}$$

- If the matrix on the left has determinant 5, what is the determinant of the matrix on the right?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{pmatrix}$$

- If the area of a fish (in a photo) is 7 square inches, and we apply a shear, what is the new area?
- Suppose that T is a linear transformation with the property that $T \circ T = T$. What is the determinant of the standard matrix for T?
- Suppose that T is a linear transformation with the property that $T \circ T = \text{identity}$. What is the determinant of the standard matrix for T?
- Find the volume of the triangular pyramid with vertices $(0,0,0)$, $(0,0,1)$, $(1,0,0)$, and $(1,2,3)$.

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.

Typical Exam Questions 4.2

- True or false. The cofactor expansion across the first row gives the negative of the cofactor expansion across the second row.
- Find the determinant of the following matrix using one of the formulas from this section:
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & 0 & 9 \end{pmatrix}$$
- Find the determinant of the following matrix using one of the formulas from this section:
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$
- Find the cofactor matrix for the above matrix and use it to find the inverse.

Summary of Section 5.1

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- Recipe: The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A
- Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations

Typical exam questions 5.1

- Find the 2-eigenvectors for the matrix

$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

- True or false: The zero vector is an eigenvector for every matrix.
- How many different eigenvalues can there be for an $n \times n$ matrix?
- Consider the reflection of $\mathbb{R}^2$ about the line $y = 7x$. What are the eigenvalues (of the standard matrix)?
- Consider the $\pi/2$ rotation of $\mathbb{R}^3$ about the z-axis. What are the eigenvalues (of the standard matrix)?

Summary of Section 5.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Typical Exam Questions 5.2

- True or false: Every $n \times n$ matrix has an eigenvalue.
- True or false: Every $n \times n$ matrix has n distinct eigenvalues.
- True or false: The nullity of $A - \lambda I$ is the dimension of the λ-eigenspace.
- What are the eigenvalues for the standard matrix for a reflection?
- What are the eigenvalues and eigenvectors for the $n \times n$ zero matrix?
- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 0 \end{pmatrix}$$

- Find the eigenvalues of the following matrix.

$$\begin{pmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & 2 \end{pmatrix}$$

Hint: All of the eigenvalues are integers. Use the rational root theorem to guess one of the eigenvalues, and then factor out a linear term.

Summary of Section 5.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Suppose we have

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

Describe the behavior of $A^n v$ for various choices of v.

Typical Exam Questions 5.4

- True or False. If A is a 3×3 matrix with eigenvalues 0, 1, and 2, then A is diagonalizable.
- True or False. It is possible for an eigenspace to be 0-dimensional.
- True or False. Diagonalizable matrices are invertible.
- True or False. Diagonal matrices are diagonalizable.
- True or False. Upper triangular matrices are diagonalizable.
- Find the 100th power of $\left(\begin{smallmatrix} 2 & 6 \\ 0 & -1 \end{smallmatrix}\right)$.
- For each of the following matrices, diagonalize or show they are not diagonalizable:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Summary of Section 5.5

- Complex numbers allow us to solve all polynomials completely, and find n eigenvalues for an $n \times n$ matrix, counting multiplicity
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$

Typical Exam Questions 5.5

- True/False. If v is an eigenvector for A with complex entries then $i \cdot v$ is also an eigenvector for A.
- True/False. If $(i, 1)$ is an eigenvector for A then $(i, -1)$ is also an eigenvector for A.
- If A is a 4×4 matrix with real entries, what are the possibilities for the number of non-real eigenvalues of A?
- Find the eigenvalues and eigenvectors for the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$