Announcements Nov 29

RECORD

- Masks $\rightsquigarrow$ Thank you! & 6.2
- WeBWorK 5.6 & 6.1 due Tue @ midnight
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Cumulative Final exam Tue Dec 14 6-8:50 pm on Teams.

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Indoor Math Lab: Mon–Thu 11–6, Fri 11–3 Clough 246 + 252
- Outdoor Math Lab: Tue–Thu 2–4 Skiles Courtyard
- Virtual Math Lab `https://tutoring.gatech.edu/drop-in/`
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

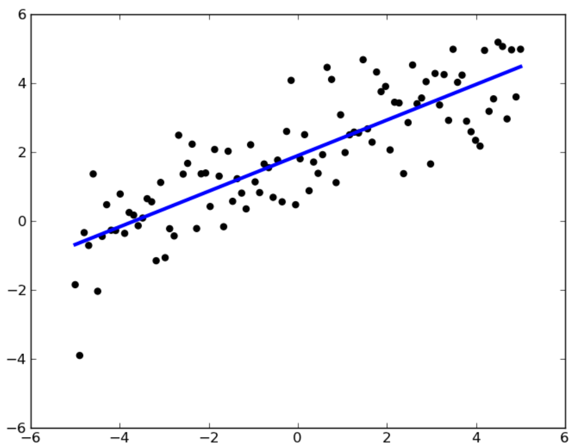
Where are we?

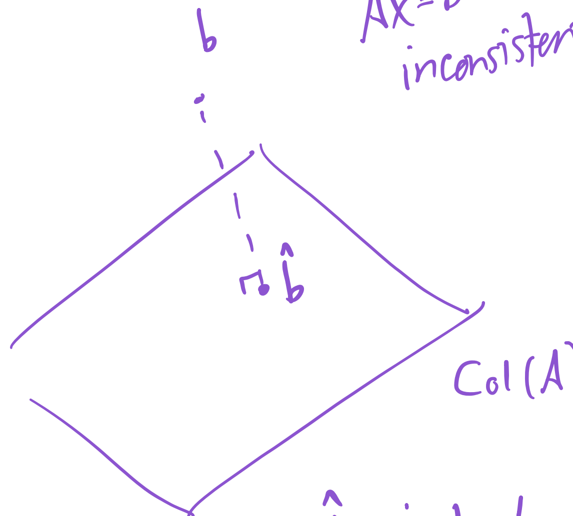
We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?

$Ax = b$ inconsistent.





dot product 0.

Solve $Ax = \hat{b}$ instead.

The answer relies on orthogonality.

Section 6.2

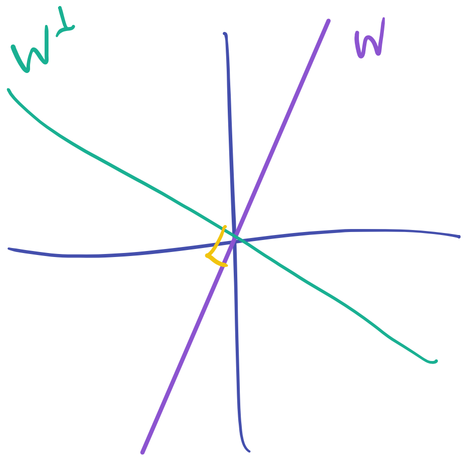
Orthogonal complements

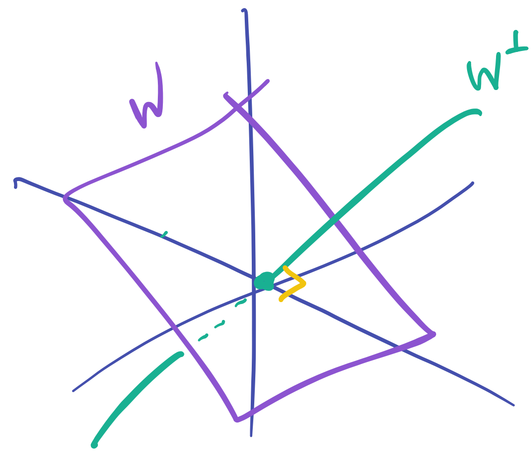
Orthogonal complements

W = subspace of $\mathbb{R}^n$

$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?
What about the orthogonal complement of a plane in $\mathbb{R}^3$?





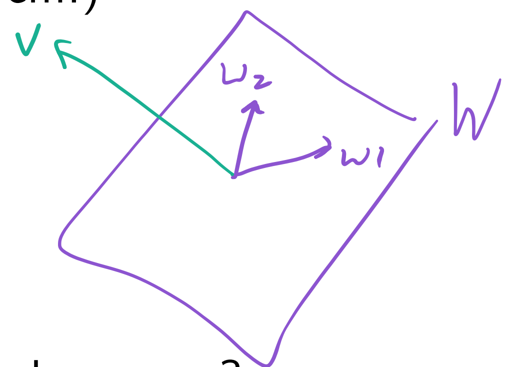
Demo Demo

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$ (it's a null space!)
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$ (rank-nullity theorem!)
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.



For items 1 and 3, which linear transformation do we use?

Orthogonal complements

Finding them

Recipe. To find (basis for) $W^\perp$, find a basis for W, make those vectors the rows of a matrix, and find (a basis for) the null space.

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

In other words:

Theorem. $A = m \times n$ matrix

$$(\mathrm{Row} A)^\perp = \mathrm{Nul}\, A$$

Geometry $\leftrightarrow$ Algebra

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

$$W^\perp = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

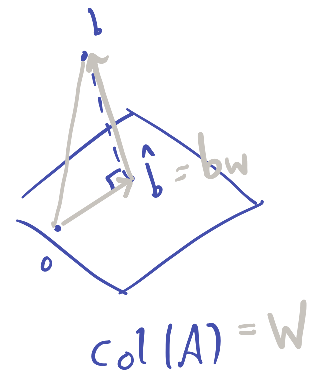
(The row space of A is the span of the rows of A.)

Orthogonal decomposition

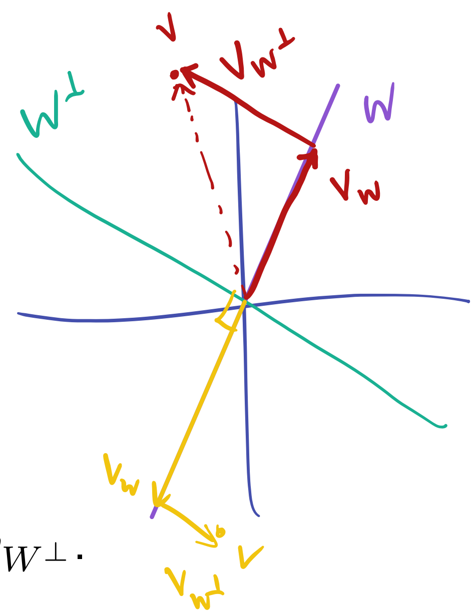
Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = v_W + v_{W^\perp}$$

where v_W is in W and $v_{W^\perp}$ is in $W^\perp$.



Why?



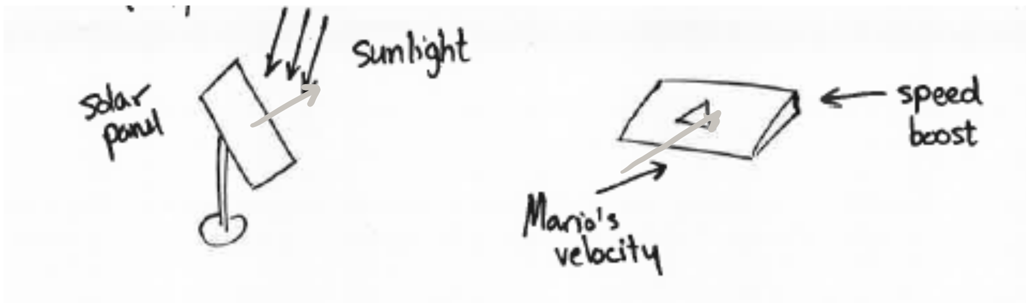
Demo

Demo

Next time: Find v_W and $v_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Also: switchbacks on hiking trails...

Section 6.3

Orthogonal projection

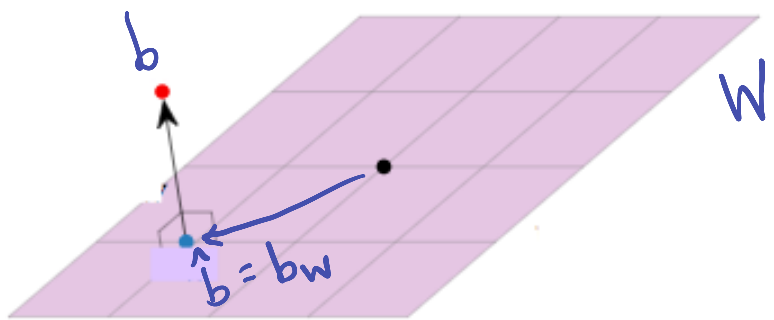
Outline of Section 6.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- Matrices for projections
- Properties of projections

Orthogonal Projections

Let b be a vector in $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.

The orthogonal projection of b onto W the vector obtained by drawing a line segment from b to W that is perpendicular to W.



Fact. The following three things are all the same:

- The orthogonal projection of b onto W
- The vector b_W (the W-part of b) algebra!
- The closest vector in W to b geometry!

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection $\overset{\hat{b}}{\overset{\shortparallel}{b_W}}$ is equal to Ax where x is any solution.

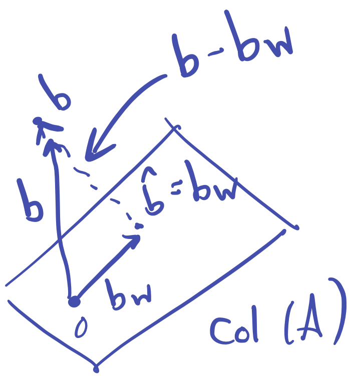
Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Why? Choose $\widehat{x}$ so that $A\widehat{x} = b_W$. We know $b - b_W = b - A\widehat{x}$ is in $W^\perp = \mathrm{Nul}(A^T)$ and so



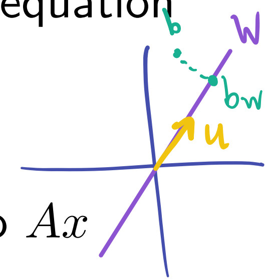
$$0 = A^T(b - A\widehat{x}) = A^T b - A^T A\widehat{x}$$

$$\rightsquigarrow A^T A\widehat{x} = A^T b$$

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$



is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

What does the theorem give when $W = \mathrm{Span}\{u\}$ is a line?

col vector.

$$A^T A x = A^T b$$

$$A = \begin{pmatrix} | \\ u \\ | \end{pmatrix}$$

$$(u \cdot u) x = u \cdot b$$

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\rightsquigarrow \quad x = \frac{u \cdot b}{u \cdot u}$$

$$A^T A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

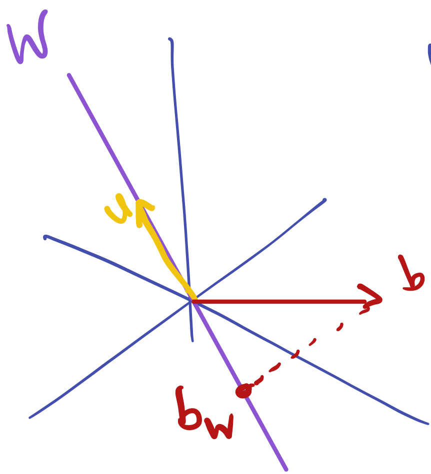
$$\rightsquigarrow \quad b_W = Ax = xu = \frac{u \cdot b}{u \cdot u} u$$

Orthogonal Projection onto a line

Special case. Let $W = \mathrm{Span}\{u\}$. For any vector b in $\mathbb{R}^n$ we have:

$$b_W = \frac{u \cdot b}{u \cdot u} u$$

Find b_W and $b_{W^\perp}$ if $b = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.



$$\frac{u \cdot b}{u \cdot u} u = \frac{-2}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -2/3 \\ -2/3 \end{pmatrix}$$

$$u \cdot b = -2 \cdot -1 + -3 \cdot 1 + -1 \cdot 1 = -2$$

$$u \cdot u = -1^2 + 1^2 + 1^2 = 3$$

Orthogonal Projections

$b_W + b_{W^\perp} = b$

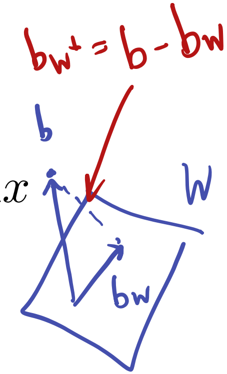
$b_{W^\perp} = b - b_W$

Same

Theorem. Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.



Example. Find b_W if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T b$, then solve for x, then compute Ax.

Question. How far is b from W? $\|b - b_W\|$

Orthogonal Projections

Example. Find b_W if $b = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find A^TA and A^Tb, then solve for x, then compute Ax.

① $A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

② $A^Tb = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$

③ $\left(\begin{array}{cc|c} 2 & 1 & 10 \\ 1 & 2 & 11 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & 1 & 10 \end{array}\right) \rightsquigarrow \begin{pmatrix} 1 & 2 & 11 \\ 0 & -3 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 11 \\ 0 & 1 & 4 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right)$

$\left(A^TA \;\middle|\; A^Tb \right)$

④ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \underbrace{\boxed{\begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}}}_{b_W}$

Question. How far is b from W?

$\|b_{W^\perp}\| = \|b - b_W\| = \left\| \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{-1^2 + 1^2 + 1^2}$

$= \sqrt{3}$

Orthogonal Projections

Same **Theorem.** Let $W = \mathrm{Col}(A)$. For any vector b in $\mathbb{R}^n$, the equation

$$A^T A x = A^T b$$

is consistent and the orthogonal projection b_W is equal to Ax where x is any solution.

Special case. If the columns of A are independent then $A^T A$ is invertible, and so

$$b_W = \underbrace{A(A^T A)^{-1} A^T}_{\text{matrix for proj to } W} b.$$

Why? The x we find tells us which linear combination of the columns of A gives us b_W. If the columns of A are independent, there's only one linear combination.

Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^T A)^{-1} A^T.$$

Why? Basically the special case from last slide

Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^T A)^{-1} A^T.$$

Example. Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$A^T A = 2 \rightsquigarrow (A^T A)^{-1} = \frac{1}{2}$$

$$A(A^TA)^{-1}A^T = \frac{1}{2} AA^T = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The proj. of $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ to W is

$$1/2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 12 \\ 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$$

Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^T A)^{-1} A^T.$$

Example. Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

you!

Projections as linear transformations

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Then

- T is a linear transformation
- $T(b) = b$ if and only if b is in W
- $T(b) = 0$ if and only if b is in $W^\perp$
- $T \circ T = T$
- The range of T is W

Properties of projection matrices

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$ (orthogonal projection). Let A be the standard matrix for T. Then

- The 1–eigenspace of A is W (unless $W = 0$)
- The 0–eigenspace of A is $W^\perp$ (unless $W = \mathbb{R}^n$)
- $A^2 = A$
- $\mathrm{Col}(A) = W$
- $\mathrm{Nul}(A) = W^\perp$
- A is diagonalizable; its diagonal matrix has m 1's & $n - m$ 0's where $m = \dim W$ (this gives another way to find A)

You can check these properties for the matrix in the last example. It would be very hard to prove these facts without any theory. But they are all easy once you know about linear transformations!

Summary of Section 6.3

- The orthogonal projection of b onto W is b_W
- b_W is the closest point in W to b.
- The distance from b to W is $\|b_{W^\perp}\|$.
- Theorem. Let $W = \mathrm{Col}(A)$. For any b, the equation $A^TAx = A^Tb$ is consistent and b_W is equal to Ax where x is any solution.
- Special case. If $W = \mathrm{Span}\{u\}$ then $b_W = \frac{u \cdot b}{u \cdot u} u$
- Special case. If the columns of A are independent then A^TA is invertible, and so $b_W = A(A^TA)^{-1}A^Tb$
- When the columns of A are independent, the standard matrix for orthogonal projection to $\mathrm{Col}(A)$ is $A(A^TA)^{-1}A^T$
- Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(b) = b_W$. Then
  - T is a linear transformation
  - etc.
- If A is the standard matrix then
  - The 1–eigenspace of A is W (unless $W = 0$)
  - etc.

Typical Exam Questions 6.3

- True/false. The solution to $A^T Ax = A^T b$ is the point in $\mathrm{Col}(A)$ that is closest to b.
- True/false. If v and w are both solutions to $A^T Ax = A^T b$ then $v - w$ is in the null space of A.
- True/false. If A has two equal columns then $A^T Ax = A^T b$ has infinitely many solutions for every b.
- Find b_W and $b_{W^\perp}$ if $b = (1, 2, 3)$ and W is the span of $(1, 2, 1)$.
- Find b_W if $b = (1, 2, 3)$ and W is the span of $(1, 2, 1)$ and $(1, 0, 1)$. Find the distance from b to W.
- Find the matrix A for orthogonal projection to the span of $(1, 2, 1)$ and $(1, 0, 1)$. What are the eigenvalues of A? What is A^{100}?