Applications of Linear Algebra

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

RECORD!

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

These relations can be represented using a matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

How does this relate to matrix transformations?

▸ Demo

Announcements Oct 4

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 2.7+2.9 & 3.1 due Tuesday nite
- Quiz 2.5-3.1 (not 2.8) Friday
- Midterm 2 Oct 20 8–9:15p

- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Midsemester Questionnaire

Comments and responses

You: WeBWorK seems to be much harder than what we see in class, which I don’t find extremely fair.

Comments and responses

You: WeBWorK seems to be much harder than what we see in class, which I don't find extremely fair.

Me: We're trying to challenge you.

Comments and responses

You: This course has been pretty challenging so far. I think the quizzes are very difficult compared to what we see in class. They are way too challenging, and I think they should instead be a general test of whether we understood what was covered that week in lecture.

Comments and responses

You: This course has been pretty challenging so far. I think the quizzes are very difficult compared to what we see in class. They are way too challenging, and I think they should instead be a general test of whether we understood what was covered that week in lecture.

Me: That's what we're aiming for, although I agree that some of the quiz questions have been on the harder side.

Comments and responses

You: Please repeat questions that others ask.

Comments and responses

You: Please repeat questions that others ask.

Me: Will try. Please remind me.

Comments and responses

You: It just seems like the WeBWorK is more computational and the quizzes are majority conceptual. The disconnect can mess me up sometimes.

Comments and responses

You: It just seems like the WeBWorK is more computational and the quizzes are majority conceptual. The disconnect can mess me up sometimes.

Me: I agree. Added more T/F to worksheets

Comments and responses

You: I wish recitation was offered online.

Comments and responses

You: I wish recitation was offered online.

Me: Is it not?

Studio M02 channel.

Comments and responses

You: Please have review sessions before exams.

Comments and responses

You: Please have review sessions before exams.

Me: We have one scheduled in class. We should also have evening review sessions

Comments and responses

You: Also, it sometimes gets hard to read his slides, especially from the back of the lecture hall, so maybe he could make the text larger on his slides or just make the slides full screen on the projector rather than just keeping it in window.

Comments and responses

You: Also, it sometimes gets hard to read his slides, especially from the back of the lecture hall, so maybe he could make the text larger on his slides or just make the slides full screen on the projector rather than just keeping it in window.

Me: Sure, please remind me.

Comments and responses

You: Would be good to have more in-depth quiz solutions.

Comments and responses

You: Would be good to have more in-depth quiz solutions.

Me: Will suggest to the course corodinator.

Comments and responses

You: Nothing really has worked well.... First exam made me sad... I don't think I am smart enough...

Comments and responses

You: Nothing really has worked well.... First exam made me sad... I don't think I am smart enough...

Me: You can absolutely do it! Please reach out to me.

Comments and responses

You: It would be much more clear to me if vector hats were included when talking about vectors.

Comments and responses

You: It would be much more clear to me if vector hats were included when talking about vectors.

Me: Oooookhhh... I thought you were talking about actual hats there.

$$\hat{v} \quad \hat{w}$$

Comments and responses

You: The classroom is really far from my dorm, so traveling is a bit of a nuisance.

Comments and responses

You: The classroom is really far from my dorm, so traveling is a bit of a nuisance.

Me: Me, too!

Comments and responses

You: I really enjoy Professor Margalit's singing and guitar skills

Comments and responses

You: I really enjoy Professor Margalit's singing and guitar skills

Me: You're damn right.

Sections 3.1

Matrix Transformations

From matrices to functions

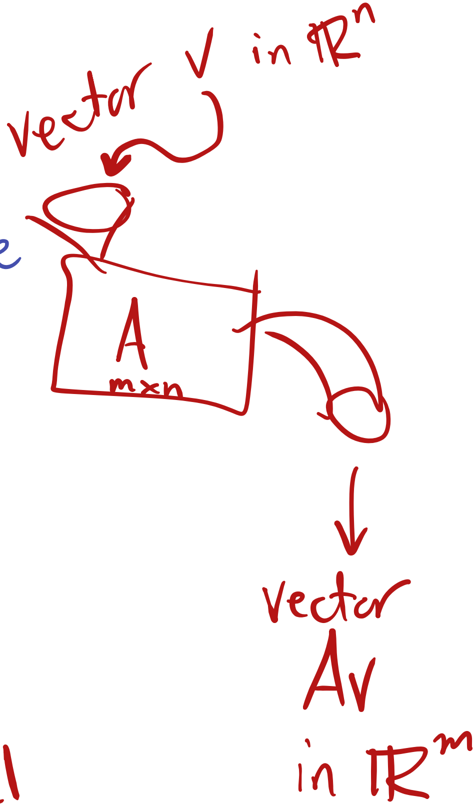
Let A be an $m \times n$ matrix.

We define a function (map)

from here to here

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(v) = Av$$



This is called a matrix transformation.

The domain of T is $\mathbb{R}^n$. inputs

The co-domain of T is $\mathbb{R}^m$. potential outputs

The range of T is the set of outputs: $\mathrm{Col}(A)$ actual atputs

This gives us a*nother* point of view of $Ax = b$

input output

▸ Demo

Why are we learning about matrix transformations?

Sample applications:

- Cryptography (Hill cypher)
- Computer graphics (Perspective projection is a linear map!)
- Aerospace (Control systems - input/output)
- Biology
- Many more!

"shear"

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Input

Output

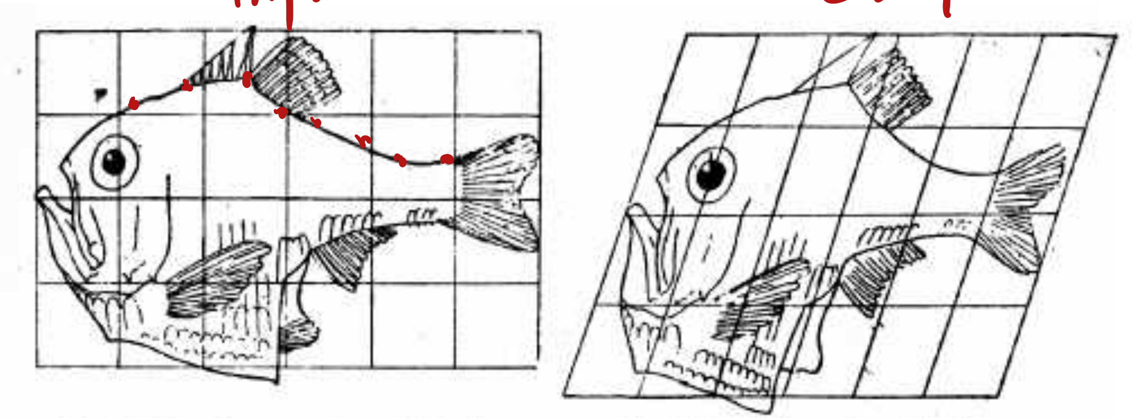


Fig. 517. *Argyropelecus Olfersi.*

Fig. 518. *Sternoptyx diaphana.*

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If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

These relations can be represented using a matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Input: population in year N

Output: population in year N+1

How does this relate to matrix transformations?

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \overset{\text{input}}{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}} = \begin{pmatrix} 28 \\ 1 \\ 1 \end{pmatrix} \leftarrow \text{output}$$

▶ Demo

Section 3.2

One-to-one and onto transformations

One-to-one and onto in calculus

What do one-to-one and onto mean for a function $f : \mathbb{R} \to \mathbb{R}$?

Examples

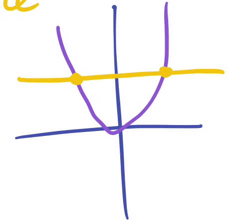
x, x^3

$3x+5, e^x$

One-to-one: Different inputs have diff outputs.

horizontal line test: each hor. line hits graph at most once

Not one-to-one: There are two inputs with same outputs



$|x|$

x^2

Example

x, x^3

$3x+5$

Onto: Range = codomain.

All vectors in codomain are outputs

each horiz line hits graph at least once.

Not onto:

x^2, e^x

$|x|$

One-to-one

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

In other words: different inputs have different outputs. Same as prev. slide.

Do not confuse this with the definition of a function, which says that for each input x in $\mathbb{R}^n$ there is ~~at most~~ exactly one output b in $\mathbb{R}^m$.

Basic examples: ① $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ all inputs have same output $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ not one-to-one

② $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ diff inputs $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}$ have diff outputs $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are linearly independent
- $Ax = 0$ has only the trivial solution ← only one input x gives the output 0.
- A has a pivot in each column
- the range of T has dimension n

We know these are same

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

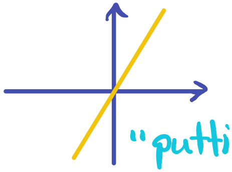
What can we say about the relative sizes of m and n if T is one-to-one?

answer: $m \geq n$ so A is tall or square.

Draw a picture of the range of a one-to-one matrix transformation $\mathbb{R} \to \mathbb{R}^{\cancel{3}2}$.

range ← col(A)

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}(3) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

"putting $\mathbb{R}^1$ inside $\mathbb{R}^2$"

Onto

A matrix transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:

- T is onto
- the columns of A span $\mathbb{R}^m$
- A has a pivot in each row
- $Ax = b$ is consistent for all b in $\mathbb{R}^m$
- the range of T has dimension m

We knew these were same

$$\mathrm{Col}(A) = \text{range } T$$

What can we say about the relative sizes of m and n if T is onto?

$m \le n$ A is wide.

Give an example of an onto matrix transformation $\mathbb{R}^3 \to \mathbb{R}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \qquad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x)$$

(projecting) squashing $\mathbb{R}^3$ to x-axis.

One-to-one and Onto

Do the following give matrix transformations that are one-to-one? onto?

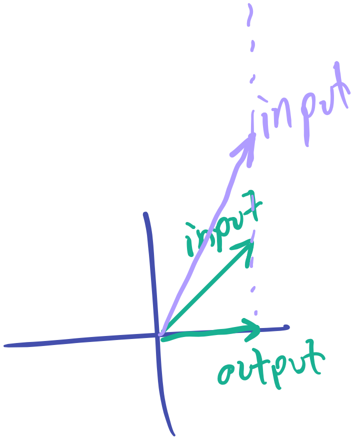
$$\begin{pmatrix} \boxed{1} & 0 & 7 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{9} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\boxed{\text{1-1}}$	✓	✓	X (wide)	✓
$\boxed{\text{onto}}$	✓	X (tall)	✓	✓

One-to-one and Onto

Which of the previously-studied matrix transformations of $\mathbb{R}^2$ are one-to-one? Onto?

		one-to-one	onto
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	reflection	✓	✓
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	projection	×	×
$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	scaling	✓	✓
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	shear	✓	✓
$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	rotation	✓	✓



Any vec. not on x-ax is not an output.

Which are one to one / onto?

Poll

Which give one to one-to-one / onto matrix transformations?

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}$$

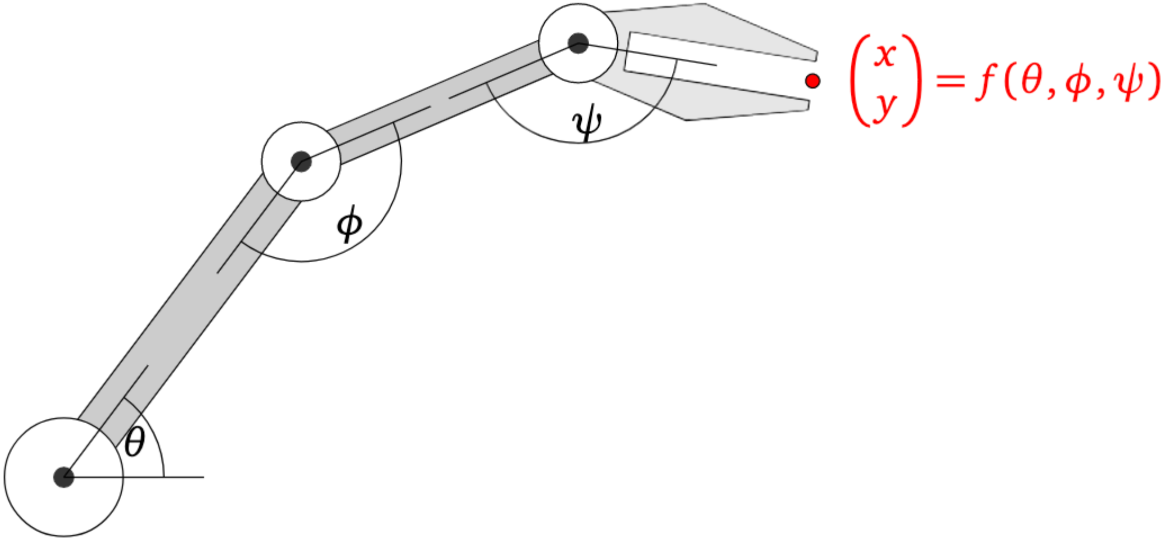
▸ Demo

▸ Demo

▸ Demo

Robot arm

Consider the robot arm example from the book.



There is a natural function f here (not a matrix transformation). The input is a set of three angles and the co-domain is $\mathbb{R}^2$. Is this function one-to-one? Onto?

The geometry

Say that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

The geometry of one-to-one:

The range has dimension n (and the null space is a point).

The geometry of onto:

The range has dimension m, so it is all of $\mathbb{R}^m$ (and the null space has dimension $n - m$).

Summary of Section 3.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the output for at most one v in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is one-to-one
  - the columns of A are linearly independent
  - $Ax = 0$ has only the trivial solution
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  - the range has dimension n
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T equals the codomain $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the output for at least one input v in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix A. Then the following are all equivalent:
  - T is onto
  - the columns of A span $\mathbb{R}^m$
  - A has a pivot in each row
  - $Ax = b$ is consistent for all b in $\mathbb{R}^m$.
  - the range of T has dimension m

Typical exam questions

- True/False. It is possible for the matrix transformation for a 5×6 matrix to be both one-to-one and onto.
- True/False. The matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by projection to the yz-plane is onto.
- True/False. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by π is onto.
- Is there an onto matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.
- Is there an one-to-one matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not.