Find a matrix that does this.
Announcements Oct 13

- Masks ☑ Thank you!
- Quiz 3.2-3.3 Friday
- WeBWorK 3.2 & 3.3 due tonite!
- Special office hr: Thu 11-12 Teams (special time!)
- Midterm 2 Oct 20 8–9:15p on Teams

- Use Piazza for general questions
- Many TA office hours listed on Canvas
- Section M web site: Google “Dan Margalit math”, click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google “Dan Margalit math”, click on Teaching
- Tutoring: http://tutoring.gatech.edu/tutoring
- PLUS sessions: http://tutoring.gatech.edu/plus-sessions
- Math Lab: http://tutoring.gatech.edu/drop-tutoring-help-desks
- Counseling center: https://counseling.gatech.edu
- You can do it!
Section 3.3

Linear Transformations
Linear transformations are matrix transformations

**Theorem.** Every linear transformation is a matrix transformation.

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ the standard matrix is:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{pmatrix}$$

**Why?** Notice that $Ae_i = T(e_i)$ for all $i$. Then it follows from linearity that $T(v) = Av$ for all $v$. 

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Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the $y$-axis and then rotates counterclockwise by $\pi/2$.

Find $T(e_1), T(e_2)$

$T(e_1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$
Find a matrix that does this.

Discussion Question

Transformation Challenge

scale by $\frac{1}{2}$ in x dir
or
rotate by $\pi/2$ clockwise
rotate $\pi/2$ clockwise
then scale by $\frac{1}{2}$
in y - dir.

$\begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix} \begin{pmatrix} -6 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
Section 3.4
Matrix Multiplication
Section 3.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things
Function composition

Remember from calculus that if $f$ and $g$ are functions then the composition $f \circ g$ is a new function defined as follows:

$$f \circ g(x) = f(g(x))$$

In words: first apply $g$, then $f$.

Example: $f(x) = x^2$ and $g(x) = x + 1$.

Note that $f \circ g$ is usually different from $g \circ f$.

$$f \circ g (x) = (x+1)^2$$

$$g \circ f (x) = x^2 + 1$$
Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^p \to \mathbb{R}^m$ and $U : \mathbb{R}^n \to \mathbb{R}^p$ and make the composition $T \circ U$.

Notice that both have an $p$. Why?

What are the domain and codomain for $T \circ U$?

Natural question: What is the matrix for $T \circ U$? We’ll see!

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?
Composition of linear transformations

Example. \( T = \) projection to \( y \)-axis and \( U = \) reflection about \( y = x \) in \( \mathbb{R}^2 \)

What is the standard matrix for \( T \circ U \)?

What about \( U \circ T \)?

\[
T \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \\
U \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

\[
T \circ U \leftrightarrow \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}
\]

Usual recipe

\[
U \circ T \leftrightarrow \begin{pmatrix} 0 & \frac{1}{2} \\ -1 & 0 \end{pmatrix}
\]
Matrix Multiplication
And now for something completely different (not really!)

Suppose $A$ is an $m \times n$ matrix. We write $a_{ij}$ or $A_{ij}$ for the $ij$th entry.

If $A$ is $m \times n$ and $B$ is $n \times p$, then $AB$ is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where $r_i$ is the $i$th row of $A$, and $b_j$ is the $j$th column of $B$.

Or: the $j$th column of $AB$ is $A$ times the $j$th column of $B$.

Multiply these matrices (both ways):
Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do $U$ then do $T$

**Fact.** Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is $AB$.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv) = (AB)v$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case $Bv$ is the $i$th column of $B$. So the left-hand side is $A$ times the $i$th column of $B$. The right-hand side is the $i$th column of $AB$ which we already said was $A$ times the $i$th column of $B$. It works!
Matrix Multiplication and Linear Transformations

Fact. Suppose that $A$ and $B$ are the standard matrices for the linear transformations $T : \mathbb{R}^{p} \to \mathbb{R}^{m}$ and $U : \mathbb{R}^{n} \to \mathbb{R}^{p}$. The standard matrix for $T \circ U$ is $AB$.

Example. $T =$ projection to $y$ axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

$$
\begin{align*}
T &\iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
U &\iff \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
$$

$$
\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}
$$

$T \circ U$ same as:

$$
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}
$$

rot. clock $\pi/2$

div. by $1/2$
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the $xy$-plane and then projects onto the $yz$-plane.

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]
Discussion Question

Are there nonzero matrices $A$ and $B$ with $AB = 0$?

1. Yes
2. No
Properties of Matrix Multiplication

- \( A(BC) = (AB)C \) \hspace{1cm} \text{assoc.}
- \( A(B + C) = AB + AC \) \hspace{1cm} \text{distrib.}
- \( (B + C)A = BA + CA \) \hspace{1cm} \text{distrib.}
- \( r(AB) = (rA)B = A(rB) \)
- \( (AB)^T = B^T A^T \)
- \( I_m A = A = A I_n \), where \( I_k \) is the \( k \times k \) identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

**Warning!**

- \( AB \) is not always equal to \( BA \)
- \( AB = AC \) does not mean that \( B = C \)
- \( AB = 0 \) does not mean that \( A \) or \( B \) is 0
More rabbits

Recall that the following matrix describes the change in our rabbit population from this year to the next:

\[
A = \begin{pmatrix}
0 & 6 & 8 \\
1/2 & 0 & 0 \\
0 & 1/2 & 0
\end{pmatrix}
\]

What matrix should we use if we want to describe the change in the rabbit population from this year to two years from now? Or 10 years from now?

\[
A^2v = \text{this year's popul.}
\]

\[
AV = \text{next year's popul.}
\]

\[
A^2V = \text{year after that}
\]

\[
A^{100}v = 100 \text{ years after 1st year}
\]
Fun with matrix multiplication

Play the Buzz game!

http://textbooks.math.gatech.edu/ila/demos/transform_game.html

In the rotation game, you need to find a composition of shears that gives a rotation!
Summary of Section 3.4

- Composition: \((T \circ U)(v) = T(U(v))\) (do \(U\) then \(T\))
- Matrix multiplication: \((AB)_{ij} = r_i \cdot b_j\)
- Matrix multiplication: the \(i\)th column of \(AB\) is \(A(b_i)\)
- Suppose that \(A\) and \(B\) are the standard matrices for the linear transformations \(T : \mathbb{R}^n \rightarrow \mathbb{R}^m\) and \(U : \mathbb{R}^p \rightarrow \mathbb{R}^n\). The standard matrix for \(T \circ U\) is \(AB\).
- **Warning!**
  - \(AB\) is not always equal to \(BA\)
  - \(AB = AC\) does not mean that \(B = C\)
  - \(AB = 0\) does not mean that \(A\) or \(B\) is 0
Typical Exam Questions 3.4

• True/False. If $A$ is a $3 \times 4$ matrix and $B$ is a $4 \times 3$ matrix, then it makes sense to multiply $A$ and $B$ in both orders.

• True/False. If it makes sense to multiply a matrix $A$ by itself, then $A$ must be a square matrix.

• True/False. If $A$ is a non-zero square matrix, then $A^2$ is a non-zero square matrix.

• True/False. If $A = -I_n$ and $B$ is an $n \times n$ matrix, then $AB = BA$.

• Find the standard matrices for the projections to the $xy$-plane and the $yz$-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?

• Find the standard matrix $A$ for projection to the $xy$-plane in $\mathbb{R}^3$. What is $A^2$?

• Find the standard matrix $A$ for reflection in the $xy$-plane in $\mathbb{R}^3$. Is there a matrix $B$ so that $AB = I_3$?
Section 3.5
Matrix Inverses
Inverses

To solve

$$Ax = b$$

we might want to “divide both sides by $A$”.

We will make sense of this...
Inverses

$A = n \times n$ matrix.

$A$ is invertible if there is a matrix $B$ with

$$AB = BA = I_n$$

$B$ is called the inverse of $A$ and is written $A^{-1}$

Example:

$$
\begin{pmatrix}
  2 & 1 \\
  1 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
  1 & -1 \\
  -1 & 2
\end{pmatrix}
$$
The $2 \times 2$ Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of $A$.

**Fact.** If $\det(A) \neq 0$ then $A$ is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then $A$ is not invertible.

**Example.** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$. 
Solving Linear Systems via Inverses

Fact. If $A$ is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$ 

Solve

$$2x + 3y + 2z = 1$$
$$x + 3z = 1$$
$$2x + 2y + 3z = 1$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$
Solving Linear Systems via Inverses

What if we change $b$?

\[
\begin{align*}
2x + 3y + 2z &= 1 \\
x + 3z &= 0 \\
2x + 2y + 3z &= 1
\end{align*}
\]

Using

\[
\begin{pmatrix}
2 & 3 & 2 \\
1 & 0 & 3 \\
2 & 2 & 3
\end{pmatrix}^{-1} =
\begin{pmatrix}
-6 & -5 & 9 \\
3 & 2 & -4 \\
2 & 2 & -3
\end{pmatrix}
\]

So finding the inverse is essentially the same as solving all $Ax = b$ equations at once (fixed $A$, varying $b$).