Announcements Oct 18

- Masks $\rightsquigarrow$ Thank you!
- Midterm 2 Wed 8–9:15p on Teams (2 channels). Sec. 2.5–3.4 (not 2.8)
- No quiz Friday
- WeBWorK 3.4 due ~~Tue @ midnight~~ tonight!
- Usual office hour: Tue 4-5 Teams (Thu office hour cancelled)
- Review Session: Joseph Cochran Tue 7–9 Skiles 230 (backup: Clough 315)
- Review session: Prof. M Wed 4:30–5:15 Howey L1

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Math Lab: Mon–Thu 11–6, Fri 11–3 Skiles Courtyard
- Section M web site: Google “Dan Margalit math”, click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google “Dan Margalit math”, click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Section 3.4

Matrix Multiplication

Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do U then do T

Fact. Suppose that A and B are the standard matrices for the linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^p \to \mathbb{R}^n$. The standard matrix for $T \circ U$ is AB.

Why?

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case Bv is the ith column of B. So the left-hand side is A times the ith column of B. The right-hand side is the ith column of AB which we already said was A times the ith column of B. It works!

Properties of Matrix Multiplication

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$
- $(AB)^T = B^T A^T$
- $I_m A = A = AI_n$, where I_k is the $k \times k$ identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- AB is not always equal to BA
- $AB = AC$ does not mean that $B = C$
- $AB = 0$ does not mean that A or B is 0

Section 3.5

Matrix Inverses

Inverses

To solve

$$Ax = b$$

we might want to "divide both sides by A".

We will make sense of this...

$5x = 35$

$x = 35/5 = 7$

Inverses

$A = n \times n$ matrix.

A is invertible if there is a matrix B with

$$AB = BA = I_n$$

I_3 : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$I_n A = A$

$I_n v = v$

B is called the inverse of A and is written A^{-1}

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The 2×2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of A.

Fact. If $\det(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then A is not invertible.

Check:

$$\frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = I_2$$

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

$\det = 1 \cdot 4 - 3 \cdot 2 = -2$

Solving Linear Systems via Inverses

Fact. If A is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

(multiply both sides by A^{-1})

Solve

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}}_{A^{-1}} \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}}_{= x}$$

But How to find A^{-1}?

Solving Linear Systems via Inverses

What if we change b?

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 0 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$

So finding the inverse is essentially the same as solving all $Ax = b$ equations at once (fixed A, varying b).

Some Facts

Say that A and B are invertible $n \times n$ matrices.

- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

$$(A\cancel{B})(\cancel{B^{-1}}A^{-1})$$

$$A I A^{-1}$$

$$AA^{-1}$$

$$I$$

What is $(ABC)^{-1}$?

$C^{-1}B^{-1}A^{-1}$

etc.

A recipe for the inverse

Only $\square$ matrices have inverses!

Suppose $A = n \times n$ matrix.

- Row reduce $(A \mid I_n)$
- If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$.
- Otherwise, A is not invertible.

Example. Find $\overset{A}{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}}^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$A \qquad I_3$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

$I_3 \qquad A^{-1}$

What if you try this on one of our 2×2 examples, such as $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$?

Why Does This Work?

First answer: we can think of the algorithm as simultaneously solving

$$Ax_1 = e_1$$
$$Ax_2 = e_2$$

and so on. But the columns of A^{-1} are $A^{-1}e_i$, which is x_i.

Do the recipe for 2×2:

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & +1 & -1 & +2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

inverse.

Matrix algebra with inverses

We saw that if $Ax = b$ and A is invertible then $x = A^{-1}b$.

We can also, for example, solve for the matrix X, assuming that

$$AX = C + DX$$

Assume that all matrices arising in the problem are $n \times n$ and invertible.

$$AX - DX = C$$

$$\cancel{X(A-D) = C}$$

$$(A-D)X = C$$

mult. both sides ON LEFT by $(A-D)^{-1}$

$$(A-D)^{-1}(A-D)X = (A-D)^{-1}C$$

$$X = (A-D)^{-1}C.$$

Scaled vectors and invertibility

Poll

Suppose that x is a nonzero vector and $Ax = 5x$. What can you say about the matrix $A - 5I$?

- it is invertible
- it is not invertible
- we cannot tell if it is invertible or not

Invertible Functions

A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a function $U : \mathbb{R}^n \to \mathbb{R}^n$, so

$$T \circ U = U \circ T = \text{identity}$$

From calculus:
$f(x) = x^3$
$g(x) = x^{1/3}$
$f \circ g(x) = x$

That is,

$$T \circ U(v) = U \circ T(v) = v \text{ for all } v \in \mathbb{R}^n$$

$T(v) = Av$

Fact. Suppose $A = n \times n$ matrix and T is the matrix transformation. Then T is invertible *as a function* if and only if A is invertible. And in this case, the standard matrix for T^{-1} is A^{-1}.

$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Example. Counterclockwise rotation by $\pi/2$.

$U =$ clockwise rotation by $\pi/2 \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example. $T =$ proj to x-axis
What is inverse?
There isn't one!

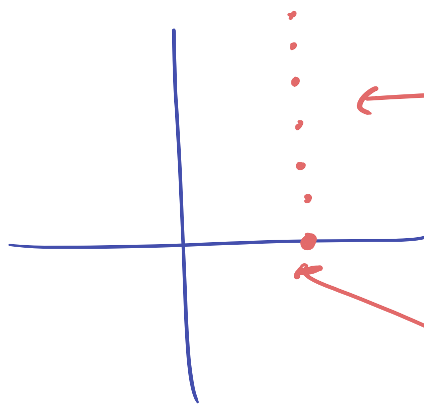
Which are invertible?

Poll

Which are invertible linear transformations of $\mathbb{R}^2$?

- reflection about the x-axis
- projection to the x-axis
- rotation by π
- reflection through the origin
- a shear
- dilation by 2

No!



all these vectors go to same place

More rabbits

$A \cdot$ current population $=$ next years population

We can use our algorithm for finding inverses to check that

$$A \quad \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1/8 & 0 & -3/2 \end{pmatrix}.$$

Recall that the first matrix tells us how our rabbit population changes from one year to the next.

If the rabbit population in a given year is $(60, 2, 3)$, what was the population in the previous year?

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1/8 & 0 & -3/2 \end{pmatrix} \begin{pmatrix} 60 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}$$

Summary of Section 3.5

- A is invertible if there is a matrix B (called the inverse) with

$$AB = BA = I_n$$

- For a 2×2 matrix A we have that A is invertible exactly when $\det(A) \neq 0$ and in this case

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- If A is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A \,|\, I_n)$.
- Invertible linear transformations correspond to invertible matrices.

Typical Exam Questions 3.5

- Find the inverse of the matrix

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

- Find a 2×2 matrix with no zeros that is equal to its own inverse.
- Solve for the matrix X. Assume that all matrices that arise are invertible:

$$C + BX = A$$

- True/False. If A is invertible, then A^2 is invertible?
- Which linear transformation is the inverse of the clockwise rotation of $\mathbb{R}^2$ by $\pi/4$?
- True/False. The inverse of an invertible linear transformation must be invertible.
- Find a matrix with no zeros that is not invertible.
- Are there two different rabbit populations that will lead to the same population in the following year?

Section 3.6

The invertible matrix theorem

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

(1) A is invertible

(2) T is invertible

(3) The reduced row echelon form of A is I_n ← RECIPE

1-1

(4) A has n pivots

already said these same

(5) $Ax = 0$ has only 0 solution   pivot every col

(6) $\mathrm{Nul}(A) = \{0\}$

(7) $\mathrm{nullity}(A) = 0$

(8) columns of A are linearly independent

(9) columns of A form a basis for $\mathbb{R}^n$

(10) T is one-to-one

onto

(11) $Ax = b$ is consistent for all b in $\mathbb{R}^n$ pivot every row

(12) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$

(13) columns of A span $\mathbb{R}^n$

(14) $\mathrm{Col}(A) = \mathbb{R}^n$

(15) $\mathrm{rank}(A) = n$

(16) T is onto

new

(17) A has a left inverse $BA = I$

(18) A has a right inverse $AB = I$

The Invertible Matrix Theorem

There are two kinds of square matrices, invertible and non-invertible matrices.

For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.

One way to think about the theorem is: there are lots of conditions equivalent to a matrix having a pivot in every row, and lots of conditions equivalent to a matrix having a pivot in every column, and when the matrix is a square, all of these many conditions become equivalent.

Example

Determine whether A is invertible. $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$

It isn't necessary to find the inverse. Instead, we may use the Invertible Matrix Theorem by checking whether we can row reduce to obtain three pivot columns, or three pivot positions.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{1} & 4 \\ 0 & 0 & \boxed{3} \end{pmatrix}$$

There are three pivot positions, so A is invertible by the IMT (statement c).

The Invertible Matrix Theorem

Poll

Which are true? Why?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity.

12

3

True

o) If A is invertible then A^2 is invertible

p) If A^2 is invertible then A is invertible

More rabbits

Discussion Question

Recall that the following matrix describes the change in our rabbit population from this year to the next:

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Which of the following statements are true?

1. There is a population of rabbits that will result in 0 rabbits in the following year.
2. There are two different populations of rabbits that result in the same population in the following year
3. For any given population of rabbits, we can choose a population of rabbits for the current year that results in the given population in the following year (this is tricky!).

Typical Exam Questions Section 3.6

In all questions, suppose that A is an $n \times n$ matrix and that $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. For each question, answer YES or NO.

(1) Suppose that the reduced row echelon form of A does not have any zero rows. Must it be true that $Ax = b$ is consistent for all b in $\mathbb{R}^n$?

(2) Suppose that T is one-to-one. Is is possible that the columns of A add up to zero?

(3) Suppose that $Ax = e_1$ is not consistent. Is it possible that T is onto?

(4) Suppose that $n = 3$ and that $T\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 0$. Is it possible that T has exactly two pivots?

(5) Suppose that $n = 3$ and that T is one-to-one. Is it possible that the range of T is a plane?