Announcements Oct 20

- Masks ~> Thank you!
- Midterm 2 Tonite! 8–9:15p on Teams (2 channels). Sec. 2.5–3.4 (not 2.8)
- No quiz Friday
- Thu office hour cancelled
- Review session: Prof. M Today 4:30–5:15 Howey L1

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Math Lab: Mon–Thu 11–6, Fri 11–3 Skiles Courtyard
- Section M web site: Google “Dan Margalit math”, click on 1553
  > future blank slides, past lecture slides, advice
- Old exams: Google “Dan Margalit math”, click on Teaching
- Tutoring: http://tutoring.gatech.edu/tutoring
- Counseling center: https://counseling.gatech.edu
- Use Piazza for general questions
- You can do it!
Review for Midterm 2
Important terms

- linearly independent
- subspace
- column space
- null space
- basis
- dimension
- one-to-one
- onto
- linear transformation
- inverse
- rank-nullity theorem
Summary of Section 2.5

- A set of vectors \( \{v_1, \ldots, v_k\} \) in \( \mathbb{R}^n \) is **linearly independent** if the vector equation

\[
x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0
\]

has only the trivial solution. It is **linearly dependent** otherwise.

- The cols of \( A \) are linearly independent

\[\iff Ax = 0 \text{ has only the trivial solution.} \iff A \text{ has a pivot in each column}\]

- The number of pivots of \( A \) equals the dimension of the span of the columns of \( A \)

- The set \( \{v_1, \ldots, v_k\} \) is linearly independent \(\iff\) they span a \( k \)-dimensional plane

- The set \( \{v_1, \ldots, v_k\} \) is linearly dependent \(\iff\) some \( v_i \) lies in the span of \( v_1, \ldots, v_{i-1} \).

- To find a collection of linearly independent vectors among the \( \{v_1, \ldots, v_k\} \), row reduce and take the (original) \( v_i \) corresponding to pivots.
Typical exam questions 2.5

• State the definition of linear independence.

• *Always/sometimes/never.* A collection of 99 vectors in $\mathbb{R}^{100}$ is linearly dependent.

• *Always/sometimes/never.* A collection of 100 vectors in $\mathbb{R}^{99}$ is linearly dependent.

• Find all values of $h$ so that the following vectors are linearly independent:

  \[
  \begin{align*}
  \{ & \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ h \end{pmatrix} \} \\
  \end{align*}
  \]

• *True/false.* If $A$ has a pivot in each column, then the rows of $A$ are linearly independent.

• *True/false.* If $u$ and $v$ are vectors in $\mathbb{R}^5$ then $\{u, v, \sqrt{2}u - \pi v\}$ is linearly independent.

• If you have a set of linearly independent vectors, and their span is a line, how many vectors are in the set?
Find \( A \) with \( \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \)

**Column Space** \( \text{Col}(A) \): \( y = 2x \)

**Null Space** \( \text{Null}(A) \): \( y = x/2 \)

\[ y = x/2 \]
\[ y - x/2 = 0 \]
\[ 2y - x = 0 \]

\[ \begin{pmatrix} 1 \\ a \end{pmatrix} \] \( \sim \) \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)

 Null space \( x + ay = 0 \)

\[ a = -2 \]
Section 2.6 Summary

- A **subspace** of \( \mathbb{R}^n \) is a subset \( V \) with:
  1. The zero vector is in \( V \).
  2. If \( u \) and \( v \) are in \( V \), then \( u + v \) is also in \( V \).
  3. If \( u \) is in \( V \) and \( c \) is in \( \mathbb{R} \), then \( cu \in V \).

- Two important subspaces: \( \text{Nul}(A) \) and \( \text{Col}(A) \)
- Find a spanning set for \( \text{Nul}(A) \) by solving \( Ax = 0 \) in vector parametric form
- Find a spanning set for \( \text{Col}(A) \) by taking pivot columns of \( A \) (not reduced \( A \))
- Four things are the same: subspaces, spans, planes through 0, null spaces

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Let \( V \) be the subset of \( \mathbb{R}^3 \) consisting of the \( x \)-axis, the \( y \)-axis, and the \( z \)-axis. Which properties of a subspace does \( V \) fail?

Find a spanning set for the plane in \( \mathbb{R}^3 \) defined by \( x + y - 2z = 0 \).
Typical exam questions

- Consider the set \( \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\} \). Is it a subspace? If not, which properties does it fail?
- Consider the \( x \)-axis in \( \mathbb{R}^3 \). Is it a subspace? If not, which properties does it fail?
- Consider the set \( \{(x, y, z, w) \in \mathbb{R}^4 \mid x + y - z + w = 0\} \). Is it a subspace? If not, which properties does it fail?
- Find spanning sets for the column space and the null space of

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}
\]

- True/False: The set of solutions to a matrix equation is always a subspace.
- True/False: The zero vector is a subspace.
\[ V = \{ (a, b) : ab > 0 \} \]

Is this a subspace?

- \begin{align*}
& \begin{pmatrix} -3 \\ 5 \end{pmatrix} \text{ in } V \\
& \begin{pmatrix} -1 \\ 5 \end{pmatrix} \text{ not in } V
\end{align*}

addition?

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

in } V \text{ in } V \text{ not in } V
Section 2.7 Summary

- **A basis** for a subspace \( V \) is a set of vectors \( \{v_1, v_2, \ldots, v_k\} \) such that
  1. \( V = \text{Span}\{v_1, \ldots, v_k\} \)
  2. \( v_1, \ldots, v_k \) are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for \( \text{Nul}(A) \) by solving \( Ax = 0 \) in vector parametric form.
- Find a basis for \( \text{Col}(A) \) by taking pivot columns of \( A \) (not reduced \( A \)).
- **Basis Theorem.** Suppose \( V \) is a \( k \)-dimensional subspace of \( \mathbb{R}^n \). Then
  - Any \( k \) linearly independent vectors in \( V \) form a basis for \( V \).
  - Any \( k \) vectors in \( V \) that span \( V \) form a basis.

\[
\begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}
\]

Find a basis \( \{u, v, w\} \) for \( \mathbb{R}^3 \) where no vector has a zero entry.

\[
\begin{pmatrix} 1 \\ 7 \\ 11 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}
\]
Typical exam questions

- Find a basis for the $yz$-plane in $\mathbb{R}^3$
- Find a basis for $\mathbb{R}^3$ where no vector has a zero
- How many vectors are there in a basis for a line in $\mathbb{R}^7$?
- True/false: every basis for a plane in $\mathbb{R}^3$ has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in $\mathbb{R}^3$ and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of $A$ is the number of pivots of $A$.
- True/false: If $b$ lies in the column space of $A$, and the columns of $A$ are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span $\mathbb{R}^3$ must be linearly independent.
Section 2.9 Summary

- Rank-Nullity Theorem. \( \text{rank}(A) + \text{dim Nul}(A) = \#\text{cols}(A) \)

Let \( A \) be an \( 4 \times 6 \) nonzero matrix and suppose the columns of \( A \) are all the same. What is \( \text{dim Nul}(A) \)?

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots \\
2 & 2 & 2 & \cdots \\
3 & 3 & 3 & \cdots \\
4 & 4 & 4 & \cdots \\
\end{pmatrix}
\]

\( \text{rank} = 1 \)

\[5\]
Typical exam questions

• Suppose that $A$ is a $5 \times 7$ matrix, and that the column space of $A$ is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.

• Suppose that $A$ is a $5 \times 7$ matrix, and that the column space of $A$ is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.

• Suppose that $A$ is a $5 \times 7$ matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?

• True/false. There is a $3 \times 2$ matrix so that the column space and the null space are both lines.

• True/false. There is a $2 \times 3$ matrix so that the column space and the null space are both lines.

• True/false. Suppose that $A$ is a $6 \times 2$ matrix and that the column space of $A$ is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some $b$ in $\mathbb{R}^6$?
Section 3.1 Summary

- If $A$ is an $m \times n$ matrix, then the associated matrix transformation $T$ is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\text{Col}(A)$.

- If $A$ is $n \times n$ then $T$ does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

Find a matrix $A$ so that the range of the matrix transformation $T(v) = Av$ is the line $y = 2x$ in $\mathbb{R}^2$. 
Typical exam questions

- What does the matrix \(
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\) do to \(\mathbb{R}^2\)?

- What does the matrix 
\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\] do to \(\mathbb{R}^2\)?

- What does the matrix 
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] do to \(\mathbb{R}^3\)?

- What does the matrix 
\[
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\] do to \(\mathbb{R}^2\)?

- True/false. If \(A\) is a matrix and \(T\) is the associated matrix transformation, then the statement \(Ax = b\) is consistent is equivalent to the statement that \(b\) is in the range of \(T\).

- True/false. There is a matrix \(A\) so that the domain of the associated matrix transformation is a line in \(\mathbb{R}^3\).
Summary of Section 3.2

- \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is **one-to-one** if each \( b \) in \( \mathbb{R}^m \) is the output for at most one \( v \) in \( \mathbb{R}^n \).

- **Theorem.** Suppose \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:
  - \( T \) is one-to-one
  - the columns of \( A \) are **linearly independent**
  - \( Ax = 0 \) has **only the trivial solution**
  - \( A \) has a pivot in every column
  - the range has dimension \( n \)

- \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is **onto** if the range of \( T \) equals the codomain \( \mathbb{R}^m \), that is, each \( b \) in \( \mathbb{R}^m \) is the output for at least one input \( v \) in \( \mathbb{R}^n \).

- **Theorem.** Suppose \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a matrix transformation with matrix \( A \). Then the following are all equivalent:
  - \( T \) is onto
  - the columns of \( A \) span \( \mathbb{R}^m \)
  - \( A \) has a pivot in every row
  - \( Ax = b \) is consistent for every \( b \) in \( \mathbb{R}^m \)
  - the range of \( T \) has dimension \( m \)

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Let \( A \) be an \( 5 \times 5 \) matrix. Suppose that \( \dim \text{Nul}(A) = 0 \). Must it be true that \( Ax = e_1 \) is consistent?

Yes, **true**.
Typical exam questions

• True/False. It is possible for the matrix transformation for a $5 \times 6$ matrix to be both one-to-one and onto.

• True/False. The matrix transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by projection to the $yz$-plane is onto. **No, Range is $yz$-plane, not all of $\mathbb{R}^3$**

• True/False. The matrix transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation by $\pi$ is onto.

• Is there an onto matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not. **No.**

• Is there an one-to-one matrix transformation $\mathbb{R}^2 \to \mathbb{R}^3$? If so, write one down, if not explain why not. **(Matrix)**
Summary of Section 3.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u + v) = T(u) + T(v)$ for all $u, v$ in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and $c$ in $\mathbb{R}$.

- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).

- The standard matrix for a linear transformation has its $i$th column equal to $T(e_i)$.

Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects over the line $y = -x$ and then rotates counterclockwise by $\pi/2$. 

$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Typical Exam Questions Section 3.3

- Is the function $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$ a linear transformation?

- Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation and that

\[
T\left(\begin{array}{c} 1 \\ 1 \end{array}\right) = \left(\begin{array}{c} 3 \\ 3 \\ 1 \end{array}\right) \quad \text{and} \quad T\left(\begin{array}{c} 2 \\ 1 \end{array}\right) = \left(\begin{array}{c} 3 \\ 1 \\ 1 \end{array}\right)
\]

What is $T\left(\begin{array}{c} 1 \\ 0 \end{array}\right)$?

- Find the matrix for the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that rotates about the $z$-axis by $\pi$ and then scales by 2.

- Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the function given by:

\[
T\left(\begin{array}{c} x \\ y \\ z \end{array}\right) = \left(\begin{array}{c} z \\ 0 \\ x \end{array}\right)
\]

Is this a linear transformation? If so, what is the standard matrix for $T$?

- Is the identity transformation one-to-one?
Summary of Section 3.4

- Composition: \((T \circ U)(v) = T(U(v))\) (do \(U\) then \(T\))
- Matrix multiplication: \((AB)_{ij} = r_i \cdot b_j\)
- Matrix multiplication: the \(i\)th column of \(AB\) is \(A(b_i)\)
- The standard matrix for a composition of linear transformations is the product of the standard matrices.
- Warning!
  - \(AB\) is not always equal to \(BA\)
  - \(AB = AC\) does not mean that \(B = C\)
  - \(AB = 0\) does not mean that \(A\) or \(B\) is 0

Find a \(2 \times 2\) matrix \(A\) so that \(A^4 = I\) and \(A^2 \neq I\).

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

Hint: Think about transformations. Rotation by \(\pi/2\)
Typical Exam Questions 3.4

- True/False. If $A$ is a $3 \times 4$ matrix and $B$ is a $4 \times 3$ matrix, then it makes sense to multiply $A$ and $B$ in both orders.
- True/False. If it makes sense to multiply a matrix $A$ by itself, then $A$ must be a square matrix.
- True/False. If $A$ is a non-zero square matrix, then $A^2$ is a non-zero square matrix.
- True/False. If $A = -I_n$ and $B$ is an $n \times n$ matrix, then $AB = BA$.
- Find the standard matrices for the projections to the $xy$-plane and the $yz$-plane in $\mathbb{R}^3$. Find the matrices for the linear transformations obtained by doing these two linear operations in the two different orders. Are the answers the same?
- Find the standard matrix $A$ for projection to the $xy$-plane in $\mathbb{R}^3$. What is $A^2$?
- Find the standard matrix $A$ for reflection in the $xy$-plane in $\mathbb{R}^3$. Is there a matrix $B$ so that $AB = I_3$?

\[
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = -I \cdot A = -I \cdot A = -A \quad \text{TRUE}
\]
A: \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

Q. Is there B so \( AB = I \)?

Yes \( B = A \)

\( A^2 = I \)
Practice 18

2b \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^k \)

- \( T(\|\|) = (\|\|) \) NO.
- For each \( x \) in \( \mathbb{R}^4 \) exactly one \( y \) in \( \mathbb{R}^k \) so \( T(x) = y \). No Function
- Every \( v \) in \( \mathbb{R}^k \) is image of at most one \( x \) in \( \mathbb{R}^4 \). This is defn of 1-1
- Range of \( T \) is 4D. Yes - pivot in each col
Midterm 2b #5

\[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]
\[ T(x) = \begin{pmatrix} 2x_1 + 2x_2 \\ -x_1 + 3x_2 \\ x_1 + x_2 \end{pmatrix} \]

Describe the \( x \)'s

So \( T(x) = 0 \).

\( \text{Nul} \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \) 2 pivots

\( \rightarrow \) pt. in \( \mathbb{R}^2 \)

Range of \( T \)

2 pivots

\( \sim \) plane.
2a #17

$T : \mathbb{R}^3 \rightarrow \mathbb{R}^7$

$T(e_1) = T(e_2)$

What is max possible dimension of range? Rank

guess: 2

\[
\begin{pmatrix}
1 & \ast & \ast \\
2 & 2 & \ast \\
3 & 3 & \ast \\
4 & 5 & 5 \\
5 & 6 & 6 \\
6 & 7 & 7 \\
\end{pmatrix}
\]
2b #15

$T : \mathbb{R}^2 \to \mathbb{R}^2$

reflect across $x$

$U : \mathbb{R}^2 \to \mathbb{R}^3$

$U(x) = \begin{pmatrix} 2x \\ 2x+y \\ 0 \end{pmatrix}$

Find std matrix. $U \circ T$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$(1 \ 0) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$
Practice 2a  \[ T(x) = Ax \] one-to-one.

one-to-one \[ \begin{cases} \text{For each } b \text{ in codom.} \\ \text{and onto} \\ T(x) = b \text{ has exactly one soln.} \end{cases} \] input
\[ V = \{(a, b) : a > 0\} \]
\[ = \{(a, b) : ab < 0\} \]
\[ = \{(a, b, c) : abc = 0\} \]

no 0
no add
no scal.
\[ u(x, y) = \left( \frac{2x - x^2}{y} \right) \]
2a # 19
Good luck!

It is not the mountain we conquer but ourselves.
- Edmund Hillary