The Invertible Matrix Theorem

Poll

Which are true? Why?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity.

o) If A is invertible then A^2 is invertible

p) If A^2 is invertible then A is invertible

Announcements Oct 25

- Masks $\rightsquigarrow$ Thank you!
- WeBWorK 3.5 & 3.6 due Tue @ midnight
- Studio but no quiz Friday
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Math Lab: Mon–Thu 11–6, Fri 11–3 Skiles Courtyard
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Section 3.6

The invertible matrix theorem

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

(1) A is invertible
(2) T is invertible
(3) The reduced row echelon form of A is I_n
(4) A has n pivots
(5) $Ax = 0$ has only 0 solution
(6) $\mathrm{Nul}(A) = \{0\}$
(7) $\mathrm{nullity}(A) = 0$
(8) columns of A are linearly independent
(9) columns of A form a basis for $\mathbb{R}^n$
(10) T is one-to-one
(11) $Ax = b$ is consistent for all b in $\mathbb{R}^n$
(12) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$
(13) columns of A span $\mathbb{R}^n$
(14) $\mathrm{Col}(A) = \mathbb{R}^n$
(15) $\mathrm{rank}(A) = n$
(16) T is onto
(17) A has a left inverse
(18) A has a right inverse

(19) A^T invertible.

T means transpose.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Span of cols of A

$=$ span of rows of A^T

The Invertible Matrix Theorem

$(AA)(A^{-1}A^{-1})$

Poll

Which are true? Why?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

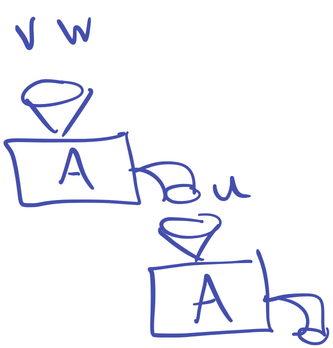
Typical n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity. Yes

o) If A is invertible then A^2 is invertible

Harder p) If A^2 is invertible then A is invertible

m) Yes dim of span of rows
= dim of span of cols
= # pivots

let me think of a better answer.



o) Yes the inverse of A^2 is: $(A^{-1})^2$

p) If A were not invertible, then T is not one-to-one then $T \circ T$ not one-to-one, so A^2 not invertible

Chapter 4

Determinants

Where are we?

- We have studied the problem $Ax = b$
- We learned to think of $Ax = b$ in terms of transformations
- We next want to study $Ax = \lambda x$ Eigenvalues
- At the end of the course we want to almost solve $Ax = b$

We need determinants for the second item.

Section 4.1

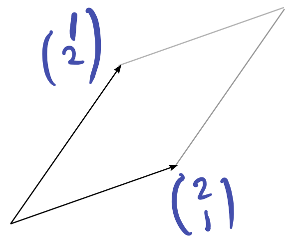
The definition of the determinant

Invertibility and volume

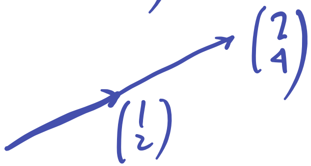
When is a 2×2 matrix invertible? $\leftarrow$ Algebra

When the rows (or columns) don't lie on a line $\Leftrightarrow$ the corresponding parallelogram has non-zero area. $\leftarrow$ Geometry

$$\det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = 1 \cdot 1 - 2 \cdot 2 = -3$$

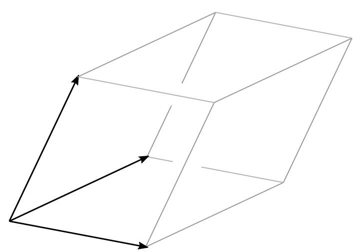


$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ one pivot



When is a 3×3 matrix invertible?

When the rows (or columns) don't lie on a plane $\Leftrightarrow$ the corresponding parallelepiped (3D parallelogram) has non-zero volume



Same for $n \times n$!

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

(signed)

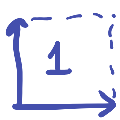
Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.

4 $I_2 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

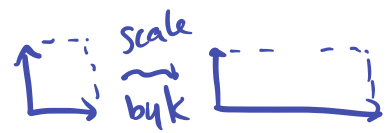
1 $\begin{pmatrix}1&0\\0&1\end{pmatrix} \xrightarrow[\text{repl}]{\text{row}} \begin{pmatrix}1&5\\0&1\end{pmatrix}$

3 $\begin{pmatrix}1&0\\0&1\end{pmatrix} \xrightarrow[\text{scale}]{\text{row}} \begin{pmatrix}3&0\\0&1\end{pmatrix}$



area 1

area doesn't change!



area scales by k.

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
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4. $\det(I_n) = 1$

Problem. Just using these rules, compute the determinants:

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

row repl

1

I

Use 1, 4

row swap

-1

I

Use 2, 4

row scale by 17

17

I

Use 3, 4

row repl

24

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ det = 24

Use 1, 4

A basic fact about determinants

Fact. If A has a zero row, then $\det(A) = 0$.

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{pmatrix} = 0 \cdot \det\begin{pmatrix} a & b & c \\ d & e & f \\ 0\cdot 0 & 0\cdot 0 & 0\cdot 0 \end{pmatrix} = 0$$

Fact. If A is a diagonal matrix then $\det(A)$ is the product of the diagonal entries.

$$\det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc$$

Fact. If A is in row echelon form then $\det(A)$ is the product of the diagonal entries.

$$\det\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix} = abc$$ (row replace to get to last fact)

Why do these follow from the definition? ✓

A first formula for the determinant

Fact. Suppose we row reduce A. Then

$$\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$$

↖ if no scalings, put a 1 here

~~Use the fact to get a formula for the determinant of any 2×2 matrix.~~

$$A \xrightarrow[\text{swap}]{\text{row}} \begin{pmatrix} \text{det } 2/35 \end{pmatrix} \xrightarrow[\text{by } 5]{\text{row scale}} \begin{pmatrix} \text{det } 2/7 \end{pmatrix} \xrightarrow[\text{swap}]{\text{row}} \begin{pmatrix} \text{det } -2/7 \end{pmatrix}$$

det $-2/35$

$$\xrightarrow[\text{by } 7]{\text{row scale}} \begin{pmatrix} \end{pmatrix} \xrightarrow[\text{repl.}]{\text{row}} \begin{pmatrix} 1 & 7 & 9 \\ 0 & 2 & 17 \\ 0 & 0 & -1 \end{pmatrix} \qquad \det A = (-1)^2 \frac{1 \cdot 2 \cdot -1}{5 \cdot 7} = -2/35$$

det -2 det $= -2$

Consequence of the above fact:

Fact. $\det A \neq 0 \Leftrightarrow A$ invertible

$$A \rightsquigarrow I \qquad \det = 1$$

Computing determinants

...using the definition in terms of row operations

$$\det\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 5 & 7 & -4\end{pmatrix} =$$

$R3 \to R3 - 5R1$

$$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 5 & 7 & -4\end{pmatrix} \xrightarrow[\text{swap}]{\text{row}} \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 5 & 7 & -4\end{pmatrix} \xrightarrow[\text{repl.}]{\text{2 row}} \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & -9\end{pmatrix} \xrightarrow[\text{by } -1/9]{\text{scale}} \begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$\boxed{\det 9}$ $\qquad \det -9 \qquad \det -9 \qquad \det 1$

or

$$(-1)^1 \frac{1\cdot 1\cdot 1}{-1/9} = 9$$

Computing determinants

...using the definition in terms of row operations

$$\det \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \xrightarrow[\text{swaps}]{\text{2 row}} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 6 & 8 \end{pmatrix} \xrightarrow[\text{repl.}]{\text{row}} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$\det\ 8 \cdot \frac{1}{2} \cdot \frac{1}{2} = 2$$

$$\boxed{\det = 2}$$

$$\det A = (-1)^2 \; \frac{1/2 \cdot 1/2 \cdot 8}{1.}$$

A Mathematical Conundrum

We have this definition of a determinant, and it gives us a way to compute it.

But: we don't know that such a determinant function exists.

More specifically, we haven't ruled out the possibility that two different row reductions might gives us two different answers for the determinant.

Don't worry! It is all okay.

We already gave the key idea: that determinant is just the volume of the corresponding parallelepiped. You can read the proof in the book if you want.

Fact 1. There is such a number $\det$ and it is unique.

Properties of the determinant

Fact 1. There is such a number $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions. Next time!

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$ ok, now we need to say what transpose is

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ $\det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - cb$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

or $|\det(A)|$ is the volume/area.

If you want the proofs, see the book. Actually Fact 1 is the hardest!

Powers

$$\det BA$$
$$\shortparallel$$

Fact 5. $\det(AB) = \det(A)\det(B)$ $\longrightarrow$ $\det(A^5) = (\det A)^5$

Use this fact to compute

$$\det\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 5 & 7 & -4 \end{pmatrix}^5\right) = 9^5$$

$$\det A \det A^{-1}$$
$$= \det(AA^{-1}) = 1$$

What is $\det(A^{-1})$? $1/9$

$$\shortparallel$$
$$(\det A)^{-1}$$

Powers

Poll

Suppose we know A^5 is invertible. Is A invertible?

1. yes
2. no
3. maybe

Section 4.3

The determinant and volumes

Areas of triangles

What is the area of the triangle in $\mathbb{R}^2$ with vertices $(1, 2)$, $(4, 3)$, and $(2, 5)$?

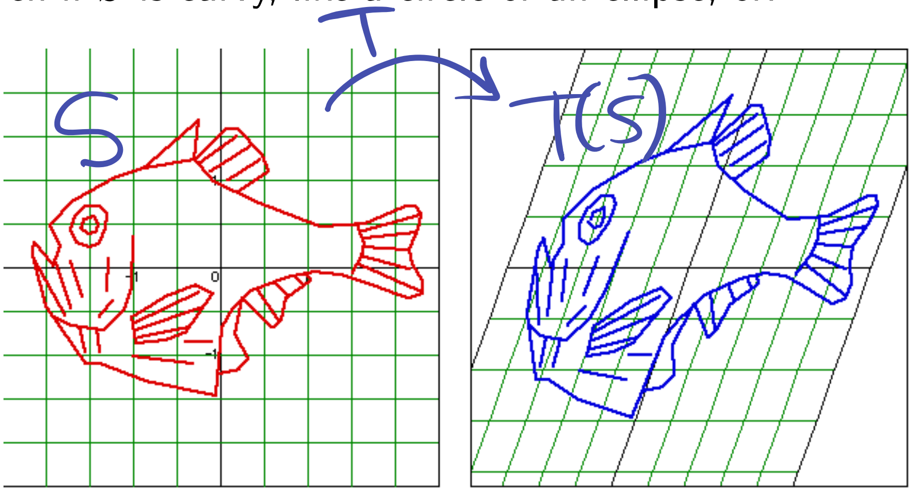
What is the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1, 2)$, $(4, 3)$, $(2, 5)$, and $(5, 6)$?

Determinants and linear transformations

Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

Summary of Sections 4.1 and 4.3

Say $\det$ is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\dfrac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

Typical Exam Questions 4.1 and 4.3

- Find the value of h that makes the determinant 0:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 2 & h \end{pmatrix}$$

- If the matrix on the left has determinant 5, what is the determinant of the matrix on the right?

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} g & h & i \\ d & e & f \\ a-d & b-e & c-f \end{pmatrix}$$

- If the area of a fish (in a photo) is 7 square inches, and we apply a shear, what is the new area?
- Suppose that T is a linear transformation with the property that $T \circ T = T$. What is the determinant of the standard matrix for T?
- Suppose that T is a linear transformation with the property that $T \circ T = \text{identity}$. What is the determinant of the standard matrix for T?
- Find the volume of the triangular pyramid with vertices $(0,0,0)$, $(0,0,1)$, $(1,0,0)$, and $(1,2,3)$.