Announcements Oct 27

- Masks $\rightsquigarrow$ Thank you!
- Studio but no quiz Friday
- WeBWorK 4 due Tue @ midnight
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Midterm 3 Nov 17 8–9:15 on Teams, Sec. 3.5–5.5

- Many TA office hours listed on Canvas
- PLUS sessions: Tue 6–7 GT Connector, Thu 6–7 BlueJeans
- Math Lab: Mon–Thu 11–6, Fri 11–3 Skiles Courtyard (going hybrid!)
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- Counseling center: `https://counseling.gatech.edu`
- Use Piazza for general questions
- You can do it!

Section 4.1

The definition of the determinant

The definition of determinant

The determinant of a *square* matrix is a number so that

1. If we do a row replacement on a matrix, the determinant is unchanged
2. If we swap two rows of a matrix, the determinant scales by -1
3. If we scale a row of a matrix by k, the determinant scales by k
4. $\det(I_n) = 1$

Why would we think of this? *Answer: This is exactly how volume works.*

Try it out for 2×2 matrices.

Today: A formula for det

like $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

Properties of the determinant

Fact 1. There is such a number $\det$ and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\dfrac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

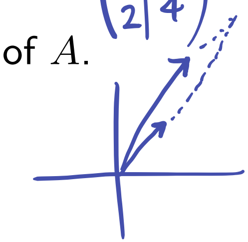
Fact 4. The function can be computed by any of the $2n$ cofactor expansions. today!

Fact 5. $\det(AB) = \det(A)\det(B)$ important! chain rule!

Fact 6. $\det(A^T) = \det(A)$ ok, now we need to say what transpose is

$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.



If you want the proofs, see the book. Actually Fact 1 is the hardest!

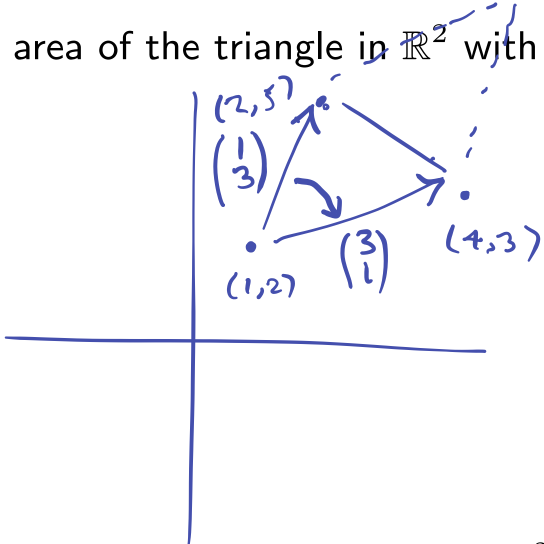
Section 4.3

The determinant and volumes

Areas of triangles

What is the area of the triangle in $\mathbb{R}^2$ with vertices $(1, 2)$, $(4, 3)$, and $(2, 5)$?

$\det\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$



$\left|\det\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\right| = |-8| = 8$

area of parallelogram

$\boxed{4}$ area of $\triangle$

What is the area of the parallelogram in $\mathbb{R}^2$ with vertices $(1, 2)$, $(4, 3)$, $(2, 5)$, and $(5, 6)$?

you!

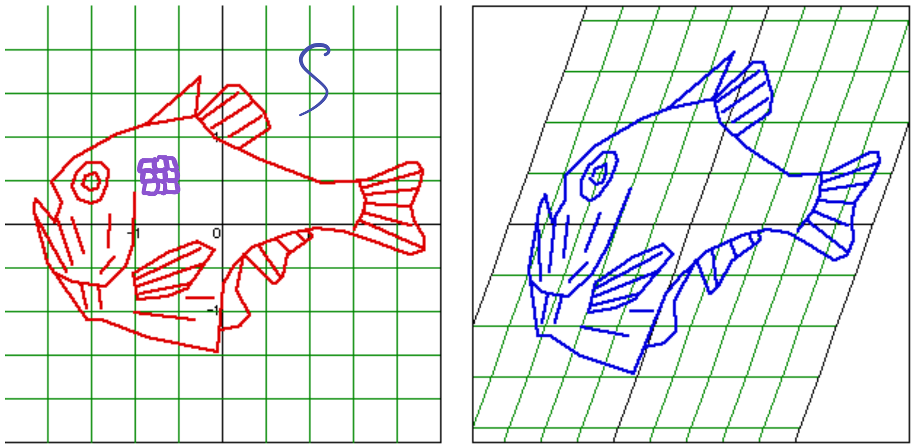
Determinants and linear transformations

apply T to every pt of S

Say A is an $n \times n$ matrix and $T(v) = Av$.

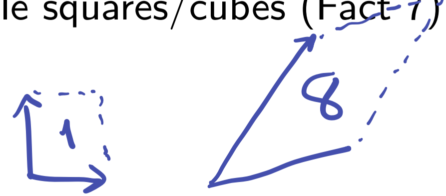
Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

This works even if S is curvy, like a circle or an ellipse, or:



Why? First check it for little squares/cubes (Fact 7). Then: Calculus!

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

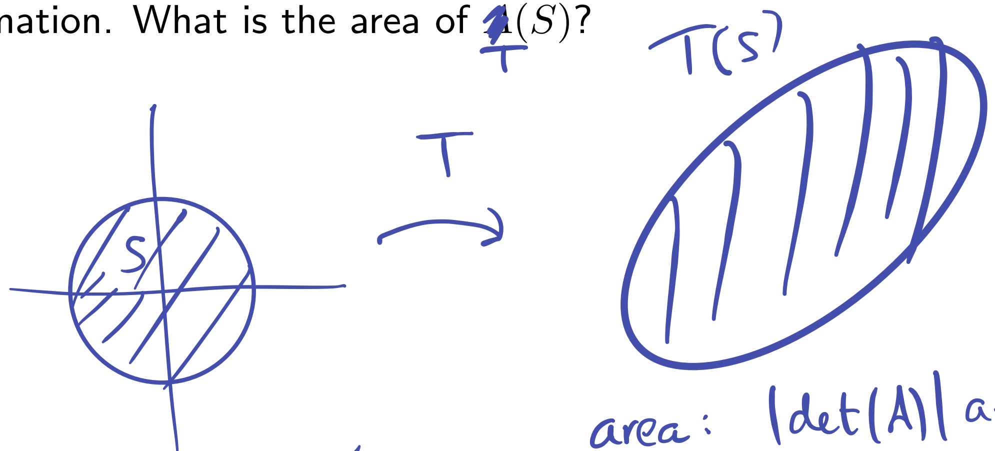


Determinants and linear transformations

Say A is an $n \times n$ matrix and $T(v) = Av$.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

Take S to be the unit disk in $\mathbb{R}^2$, that is, the set of points that have distance at most 1 from the origin. Let $A = \left(\begin{smallmatrix} 2 & 2 \\ 4 & 2 \end{smallmatrix}\right)$, and let $T(v) = Av$ be its matrix transformation. What is the area of $\cancel{A}(S)$?
T



area $\pi \cdot 1^2 = \pi$

area: $|\det(A)|\, \text{area}(S)$

$|4-8|\,\pi$

$= 4\pi$

Section 4.2

Cofactor expansions

Outline of Section 4.2

- We will give a recursive formula for the determinant of a square matrix.

A formula for the determinant

We will give a recursive formula.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$

First some terminology:

$$A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

A_{ij} = ijth minor of A
= $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column

$C_{ij} = (-1)^{i+j} \det(A_{ij})$
= ijth cofactor of A

$$C_{11} = (-1)^{1+1} \det A_{11} = 1 \cdot -3 = -3$$
$$C_{12} = (-1)^{1+2} \det A_{12} = -1 \cdot -6 = 6$$
$$C_{13} = (-1)^{1+3} \det A_{13} = 1 \cdot -3 = -3$$

Finally: entry of A in 1st row 1st col — all 1's

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

$$1 \cdot -3 + 2 \cdot 6 + 3 \cdot -3 = 0$$

(not invertible!)

Or:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

So we find the determinant of a 3×3 matrix in terms of the determinants of 2×2 matrices, etc.

Determinants

Consider

$$A = \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

Compute the following:

$a_{11} = 5$ $\qquad a_{12} = 1$ $\qquad a_{13} = 0$

$A_{11} = \begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix}$ $\qquad A_{12} = \begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix}$ $\qquad A_{13} = \begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$

$\det A_{11} = -3$ $\qquad \det A_{12} = -7$ $\qquad \det A_{13} = -12$

$C_{11} = -3$ $\qquad C_{12} = +7$ $\qquad C_{13} = -12$

$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ $\quad 5 \cdot -3 + 1 \cdot 7 + 0 \cdot$ don't care $= -8$

A formula for the determinant

We can take the recursive formula further....

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

Say that....

1×1 matrices

$$\det(a_{11}) = a_{11}$$

Now apply the formula to...

2×2 matrices

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \quad a_{11}a_{22} - a_{12}a_{21}$$

(Could also go really nuts and define the determinant of a 0×0 matrix to be 1 and use the formula to get the formula for 1×1 matrices...)

A formula for the determinant

3×3 matrices

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \cdots$$

You can write this out. And it is a good exercise. But you won't want to memorize it.

A formula for the determinant

Another formula for 3×3 matrices

original cofactor formula.

$a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(\quad) + a_{13}(\quad)$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

(Check this is gives the same answer as before. It is a small miracle!)

This slide ONLY 3×3

Use this formula to compute

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$\begin{matrix} & 0 & 1 & 0 & & & \\ 5 & 1 & 0 & 5 & 1 & 0 \\ -1 & 3 & 2 & -1 & 3 & 2 \\ 4 & 0 & -1 & 4 & 0 & -1 \\ & -15 & 8 & 0 & & \end{matrix}$$

$(-15+8+0)-(0+1+0)$

$= -7-1$

$= -8$

Expanding across other rows and columns

The formula we gave for $\det(A)$ is the expansion across the first row. It turns out you can compute the determinant by expanding across any row or column:

$$\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in} \text{ for any fixed } i$$
$$\det(A) = a_{1j}C_{1j} + \cdots + a_{nj}C_{nj} \text{ for any fixed } j$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Or for odd rows and columns:

$$\det(A) = a_{i1}(\det(A_{i1})) - a_{i2}(\det(A_{i2})) + \cdots \pm a_{in}(\det(A_{in}))$$
$$\det(A) = a_{1j}(\det(A_{1j})) - a_{2j}(\det(A_{2j})) + \cdots \pm a_{nj}(\det(A_{nj}))$$

and for even rows and columns:

$$\det(A) = -a_{i1}(\det(A_{i1})) + a_{i2}(\det(A_{i2})) + \cdots \mp a_{in}(\det(A_{in}))$$
$$\det(A) = -a_{1j}(\det(A_{1j})) + a_{2j}(\det(A_{2j})) + \cdots \mp a_{nj}(\det(A_{nj}))$$

Amazingly, these are all the same!

Compute:

$$\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 5 & 9 & 1 \end{pmatrix}$$

3rd col: $0 \cdot$ don't care $- 0 \cdot$ don't care $+ 1 \cdot \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1 \cdot 1 = 1.$

2nd row: $-1 \cdot 1 + 1 \cdot 2 + 0 = 1$

Another!

$$\det \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} =$$

$(-1)^{i+j}$

$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

3rd col $\quad + 8 \cdot \frac{1}{4} = 2.$

2nd row $\quad -\frac{1}{2} \cdot \det \begin{pmatrix} 6 & 8 \\ 1/2 & 0 \end{pmatrix} = -1/2 \cdot -4 = 2$

Determinants of triangular matrices

If A is upper (or lower) triangular, $\det(A)$ is easy to compute with cofactor expansions (it was also easy using the definition of the determinant):

$$\det\begin{pmatrix} 2 & 1 & 5 & -2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$\begin{pmatrix} + & - & \\ - & + & \\ & & \end{pmatrix}$$

1^{st} col: $+2 \cdot \det\begin{pmatrix} 1 & 2 & -3 \\ 0 & 5 & 9 \\ 0 & 0 & 10 \end{pmatrix}$

$= +2 \cdot 1 \cdot \det\begin{pmatrix} 5 & 9 \\ 0 & 10 \end{pmatrix}$

$= +2 \cdot 1 \cdot 50 = 100$

Determinants

Poll

What is the determinant?

$$\det \begin{pmatrix} 4 & 7 & 0 & 9 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 5 & 9 & 2 & 10 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

A formula for the inverse
(from Section 3.3)

2×2 matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \rightsquigarrow \quad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$n \times n$ matrices

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix}$$

$$= \frac{1}{\det(A)}(C_{ij})^T$$

matrix of cofactors　　transpose　　mult. by 1/det

Check that these agree! $\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 2 \\ & & \\ & & \end{pmatrix}$ $\begin{pmatrix} 0 & & \\ 0 & & \\ 2 & & \end{pmatrix}$ $\frac{1}{2}$ (last thing)

The proof uses Cramer's rule (see the notes on the course home page. We're not testing on this - it's just for your information.)

Summary of Section 4.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.

Typical Exam Questions 4.2

- True or false. The cofactor expansion across the first row gives the negative of the cofactor expansion across the second row.
- Find the determinant of the following matrix using one of the formulas from this section:
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & 0 & 9 \end{pmatrix}$$
- Find the determinant of the following matrix using one of the formulas from this section:
$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$$
- Find the cofactor matrix for the above matrix and use it to find the inverse.