Reduced Row Echelon Form

Poll

Which are in reduced row echelon form?

$$\left(\begin{array}{c|c} 1 & 0 \\ 0 & 2 \end{array}\right) \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right) \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

REF:

1. all zero rows are at the bottom, and
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above.

RREF:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

Announcements Sep 1

- Please turn on your camera if you are able and comfortable doing so
- Quiz on 1.1 Friday. Open 6:30a–8p on Canvas/Assignments, 15 mins
- WeBWorK 1.2 & 1.3 due ~~Tuesday~~ Thu nite!
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Many, many TA office hours listed on Canvas
- Studio this Friday online; Studio for M02 will be recorded/streamed
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, old quizzes/exams
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Section 1.2

Row reduction

Row Reduction Algorithm

To find row echelon form:

Step 1 Swap rows so a leftmost nonzero entry is in 1st row (if needed)

Step 2 Scale 1st row so that its leading entry is equal to 1

Step 3 Use row replacement so all entries below this 1 (or, pivot) are 0

Then cover the first row and repeat the three steps.

To then find reduced row echelon form:

- Use row replacement so that all entries above the pivots are 0.

Examples.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \left(\begin{array}{ccc|c} 0 & 7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right) \left(\begin{array}{ccc|c} 4 & -5 & 3 & 2 \\ 1 & -1 & -2 & -6 \\ 4 & -4 & -14 & 18 \end{array}\right)$$

Interactive Row Reducer

Solutions of Linear Systems: Consistency

Solve the linear system associated to:

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 5 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

=

Pivot on RHS

$\Longleftrightarrow$ inconsistent

Say the variables are x, y, and z.

$0 = 1$

A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.

1.3 Parametric Form

Outline of Section 1.3

- Find the parametric form for the solutions to a system of linear equations.
- Describe the geometric picture of the set of solutions.

Free Variables

We know how to understand the solution to a system of linear equations when every column to the left of the vertical line has a pivot. For instance:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

If the variables are x and y what are the solutions?

$x = 5$

$y = 2$

Free Variables

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? For instance:

no pivot in x_3 column.

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 5 & 0 \\ 0 & \boxed{1} & 2 & 1 \end{array}\right)$$

pivot: for matrix in REF, first non-0 entry in each row.

represents two equations:

$$\begin{aligned} x_1 \qquad\quad + 5x_3 &= 0 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

There is one free variable x_3, corresponding to the non-pivot column. To solve, we move the free variable to the right:

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= 1 - 2x_3 \\ x_3 &= x_3 \text{ (free; any real number)} \end{aligned}$$

∞ many solns.

This is the parametric solution. We can also write the solution as:

1 free var

3 total vars

$$(-5x_3, 1 - 2x_3, x_3)$$

If you choose $x_3 = 4$ get: $(-20, -7, 4)$

line in $\mathbb{R}^3$

What is one particular solution? What does the set of solutions look like?

Free Variables

Solve the system of linear equations in x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 \qquad + 5x_3 \qquad &= 0 \\ x_4 &= 0 \end{aligned}$$

Williams example $\left(\begin{array}{ccccc|c} 1 & 0 & 5 & 6 & 4 & 7 \\ 0 & 1 & 7 & 2 & 3 & 8 \end{array}\right)$

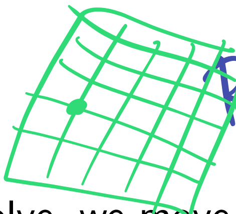
solution: 3 dim plane in $\mathbb{R}^5$

$x_1 + 5x_3 = 0$

$x_4 = 0$

So the associated matrix is:

no pivot $\rightsquigarrow$ free vars



RREF!

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right)$$

free variable = parameter

To solve, we move the free variable to the right:

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= x_2 \quad \text{(free)} \\ x_3 &= x_3 \quad \text{(free)} \\ x_4 &= 0 \end{aligned}$$

2 free vars

4 total vars

Or: $(-5x_3, x_2, x_3, 0)$. This is a plane in $\mathbb{R}^4$.

These are all solutions. One solution is: $(-5, 2, 1, 0)$ $x_3 = 1$, $x_2 = 2$

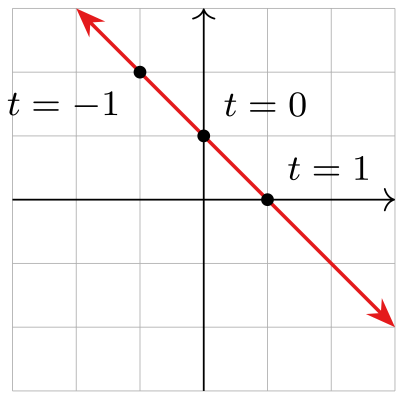
The original equations are the implicit equations for the solution. The answer to this question is the parametric solution.

Free variables

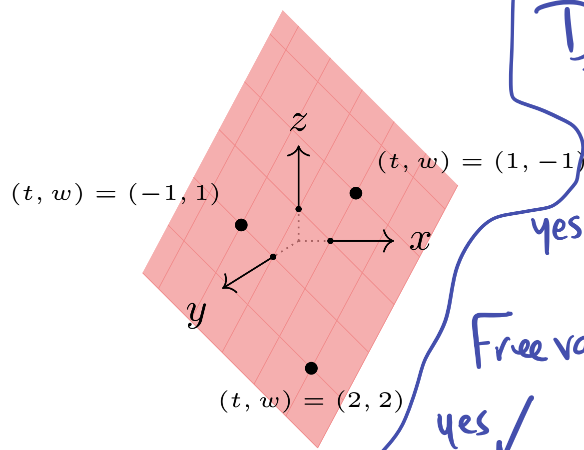
Geometry

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional plane in $\mathbb{R}^n$.

Why does this make sense?



$x+y+z=1$



Decision tree

Consistent?

yes → Free vars?

- yes → ∞ many solns, k-dim plane where k=# free vars.
- no → 1 soln

no → 0 solns

Inconsistent $\Longleftrightarrow$ pivot on right hand side

free vars = # cols on left hand side w/o pivots.

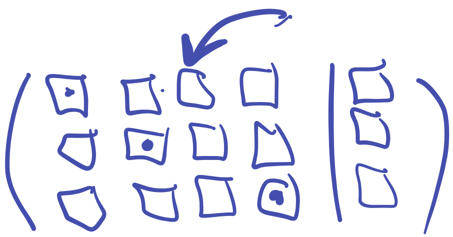
Poll

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. ~~two points~~
4. line
5. plane
6. 3-dimensional plane
7. 4-dimensional plane $= \mathbb{R}^4$

$x+y+z+w=1$
$x+y+z+w=0$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right)$$



3×5

\# pivots: 3 or 2 or 1 or 0

3 → line
2 → plane
1 → 3D plane
0 → 4D plane $= \mathbb{R}^4$

\# nonpivots

1 → line
2 → plane
3 → 3D plane
4 → 4D plane

Implicit versus parametric equations of planes

Find a parametric description of the plane

free

$$x + y + z = 1$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \end{array}\right)$$

$$x = 1 - y - z$$
$$y = y$$
$$z = z$$

$$(1 - y - z, y, z)$$

The original version is the implicit equation for the plane. The answer to this problem is the parametric description.

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of system of linear equations.

implicit

$x+y+z=1$

$x+3y-z=5$

explicit / parametric

$(1-z, 3+5z, z)$

1. The last column is a pivot column.
   $\rightsquigarrow$ the system is *inconsistent*.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   $\rightsquigarrow$ the system has a *unique solution*.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   $\rightsquigarrow$ the system has *infinitely many* solutions; free variables correspond to columns without pivots.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Typical exam questions

True/False: If a system of equations has 100 variables and 70 equations, then there must be infinitely many solutions.

True/False: If a system of equations has 70 variables and 100 equations, then it must be inconsistent.

How can we tell if an augmented matrix corresponds to a consistent system of linear equations?

If a system of linear equations has finitely many solutions, what are the possible numbers of solutions?