Announcements Sep 13

- Please wear a mask. FOR EXTRA CREDIT
- Quiz 2.1 & 2.2 Friday. Open 6:30a–8p on Canvas/Assignments, 15 mins
- WeBWorK 2.1 & 2.2 due Tuesday nite
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, old quizzes/exams, advice
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

M-Th 11-6
F 11-3

Section 2.1

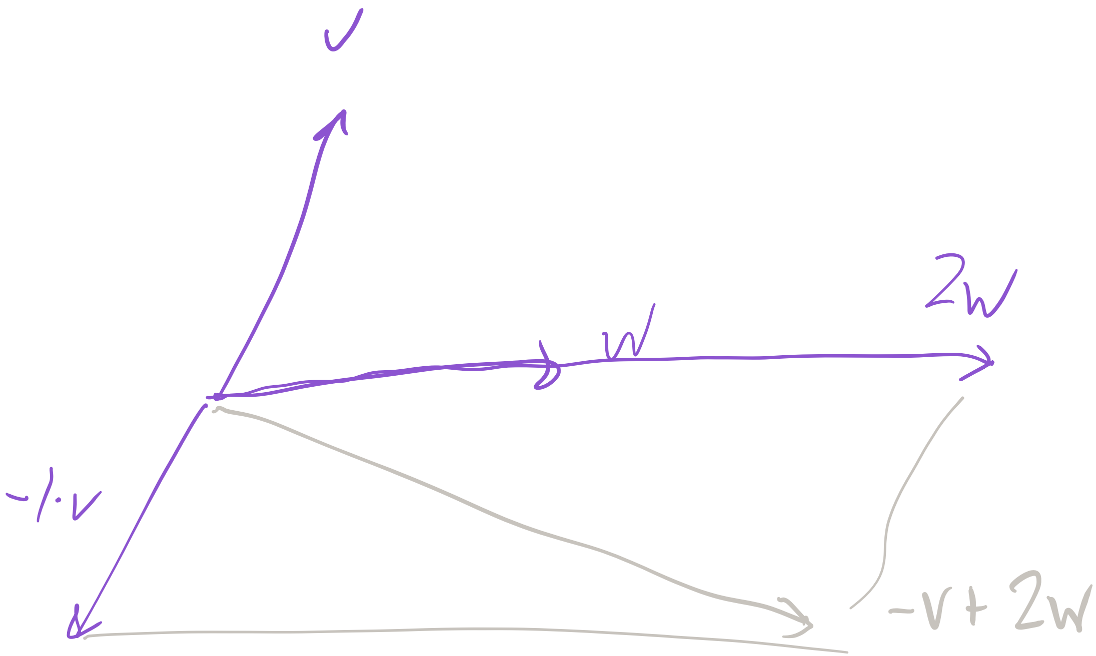
Vectors

Linear Combinations

A linear combination of the vectors $v_1, \ldots, v_k$ is any vector

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

where $c_1, \ldots, c_k$ are real numbers.



Span

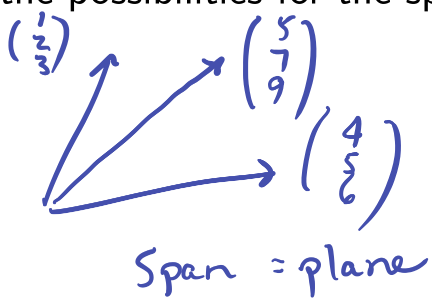
Essential vocabulary word!

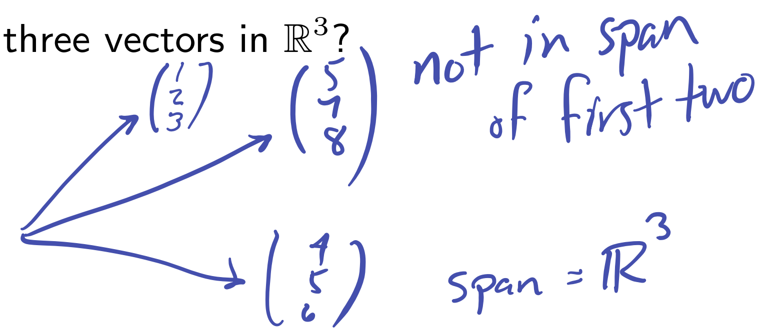
set of ... so that

$$\text{Span}\{v_1, v_2, \ldots, v_k\} = \{x_1v_1 + x_2v_2 + \cdots + x_kv_k \mid x_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)}$$
$$= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k$$
$$= \text{plane through the origin and } v_1, v_2, \ldots, v_k.$$

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo





Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

Section 2.2

Vector Equations and Spans

The four ways

Four ways of saying the same thing:

- b is in Span$\{v_1, v_2, \ldots, v_k\}$ $\leftarrow$ geometry
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $x_1 v_1 + \cdots + x_k v_k = b$ has a solution $\leftarrow$ algebra
- the system of linear equations corresponding to

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

  is consistent.

A

- $Ax = b$ has a soln

Demo

Demo

Section 2.3

Matrix equations

Multiplying Matrices by column vectors

numbers

$$\text{matrix} \times \text{column} : \begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = b_1 \begin{pmatrix} | \\ x_1 \\ | \end{pmatrix} + \cdots + b_n \begin{pmatrix} | \\ x_n \\ | \end{pmatrix}$$

columns

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = 7 \cdot \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 8 \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 21 \\ 35 \end{pmatrix} + \begin{pmatrix} 16 \\ 32 \\ 48 \end{pmatrix} = \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}$$

Multiplying Matrices by column vectors

Another way to multiply

$$\text{row vector} \times \text{column vector} : \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$$

numbers

example.

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 4$$
$$= 11$$

$$\text{matrix} \times \text{column vector} : \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

as above

rows

col vector.

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \cdot 7 + 2 \cdot 8 \\ 3 \cdot 7 + 4 \cdot 8 \\ 5 \cdot 7 + 6 \cdot 8 \end{pmatrix} = \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}$$

Linear Systems vs Augmented Matrices vs Matrix Equations vs Vector Equations

A matrix equation is an equation $Ax = b$ where A is a matrix and b is a vector. So x is a vector of variables.

A is an $m \times n$ matrix if it has m rows and n columns. What sizes must x and b be?

Solving is answering: Is $\begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$ in span of $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$?

Example:

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}_{3\times \cancel{2}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\cancel{2}\times 1} = \underbrace{\begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}}_{3\times 1}$$

matrix eqn

Rewrite this equation as a vector equation, a system of linear equations, and an augmented matrix.

vector eqn

$$x\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$$

linsys

$$x + 2y = 9$$
$$3x + 4y = 10$$
$$5x + 6y = 11$$

aug mat

you do.

We will go back and forth between these four points of view over and over again. You need to get comfortable with this.

Solving matrix equations

Solve the matrix equation

# of babies for 2nd & 3rd year rabbits

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 20 \\ 1 \\ 1 \end{pmatrix}$$

current population

$$\left(\begin{array}{ccc|c} 0 & 6 & 8 & 20 \\ 1/2 & 0 & 0 & 1 \\ 0 & 1/2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 6 & 8 & 20 \end{array}\right) \xrightarrow{R3 \to R3 - 6R2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 8 & 8 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Solution: $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

population the year before

What does this mean about rabbits?

They are so cute!

Solutions to Linear Systems vs Spans

columns of A

Say that

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

algebra $\iff$ geometry

Why?

Look back at "The Four Ways" slide.

Again this is a basic fact we will use over and over and over.

Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

last slide.

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{5} \end{array}\right)$$

inconsistent

So:

$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ not in span of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

solution: $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

So $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ is in span of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Is a given vector in the span?

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

algebra $\iff$ geometry

Is $(9, 10, 11)$ in the span of $(1, 3, 5)$ and $(2, 4, 6)$?

$$\left(\begin{array}{cc|c} 1 & 2 & 9 \\ 3 & 4 & 10 \\ 5 & 6 & 11 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & -2 & -17 \\ 0 & -4 & -34 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} \boxed{1} & 2 & 9 \\ 0 & \boxed{-2} & -17 \\ 0 & 0 & 0 \end{array}\right)$$

yes – no pivot on RHS.

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -8 \\ 0 & 1 & 17/2 \\ 0 & 0 & 0 \end{array}\right)$$

So: $-8\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + 17/2 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 9 \\ 16 \\ 11 \end{pmatrix}$

soln: $\begin{pmatrix} -8 \\ 17/2 \end{pmatrix}$

Pivots vs Solutions

Rephrasing what we know, with the "all"

Theorem. Let A be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b
2. The span of the columns of A is $\mathbb{R}^m$
3. A has a pivot in each row

Why?

(so never a pivot on RHS)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} x = b$$

has a soln when b is xy-plane.

More generally, if you have some vectors and you want to know the dimension of the span, you should row reduce and count the number of pivots.

Properties of the Matrix Product Ax

$c =$ real number, $u, v =$ vectors,

- $A(u+v) = Au + Av$
- $A(cv) = cAv$

Application. If u and v are solutions to $Ax = 0$ then so is every element of $\mathrm{Span}\{u, v\}$.

Guiding questions

Here are the guiding questions for the rest of the chapter:

1. What are the solutions to $Ax = 0$?
2. For which b is $Ax = b$ consistent?

These are two separate questions!

Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 10, 20)$, $(0, -1, -2)$
2. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 1, 0)$, $(0, 0, \sqrt{2})$
3. $(0, 1, 2)$ is in the span of $(3, 3, 4)$, $(0, 5, 7)$, $(0, 6, 8)$
4. $(0, 1, 2)$ is in the span of $(5, 7, 0)$, $(6, 8, 0)$, $(3, 3, 4)$

Summary of Section 2.3

- Two ways to multiply a matrix times a column vector:

$$\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

  OR

$$\begin{pmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_{\color{red}n} \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_{\color{red}n} \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 x_1 & \cdots & b_n x_n \\ | & & | \end{pmatrix}$$

- Linear systems, augmented matrices, vector equations, and matrix equations are all equivalent.
- Fact. $Ax = b$ has a solution $\Leftrightarrow$ b is in the span of columns of A
- Theorem. Let A be an $m \times n$ matrix. The following are equivalent.
  1. $Ax = b$ has a solution for all b
  2. The span of the columns of A is $\mathbb{R}^m$
  3. A has a pivot in each row

Typical exam questions

- If A is a 3×5 matrix, and the product Ax makes sense, then which $\mathbb{R}^n$ does x lie in?

- Rewrite the following linear system as a matrix equation and a vector equation:

$$x + y + z = 1$$

- Multiply:

$$\begin{pmatrix} 0 & 2 \\ 0 & 4 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

- Which of the following matrix equations are consistent?

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

(And can you do it without row reducing?)