# Announcements Sep 20

- Masks $\rightsquigarrow$ extra credit.
- WeBWorK 2.3 & 2.4 due Tuesday nite
- Midterm Wednesday 8–9:15p on Teams ~~in your Studio channel~~, up to Sec 2.4
- No quiz Friday
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, old quizzes/exams, advice
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

# Section 2.4

## Solution Sets

# Homogeneous vs. Nonhomogeneous Systems

*Key realization.* Set of solutions to $Ax = b$ obtained by taking one solution and adding all possible solutions to $Ax = 0$.

$$Ax = 0 \text{ solutions} \rightsquigarrow Ax = b \text{ solutions}$$

$$x_k v_k + \cdots + x_n v_n \rightsquigarrow p + x_k v_k + \cdots + x_n v_n$$

So: set of solutions to $Ax = b$ is parallel to the set of solutions to $Ax = 0$. It is a translate of a plane through the origin. (Again, we are using geometry to understand algebra!)

So by understanding $Ax = 0$ we gain understanding of $Ax = b$ for all $b$. This gives structure to the set of equations $Ax = b$ for all $b$.

Demo

Demo

# Parametric Vector Forms for Solutions

## Nonhomogeneous case

homog.

Find the parametric vector forms for $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{matrix} x_1 = 3x_2 \\ x_2 = x_2 \end{matrix} \rightsquigarrow x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

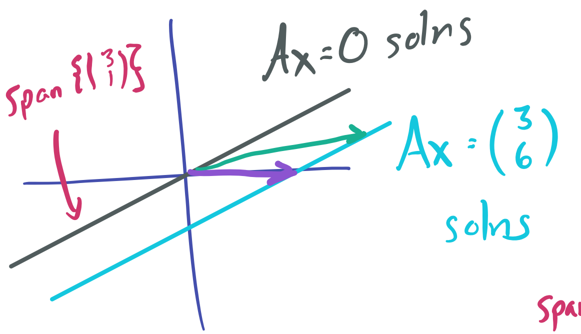
param form

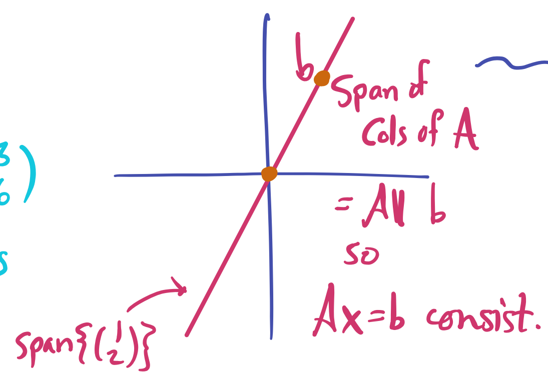
param vect form

non-homog

...and $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

$$\left(\begin{array}{cc|c} 1 & -3 & 3 \\ 2 & -6 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow \begin{matrix} x_1 = 3 + 3x_2 \\ x_2 = x_2 \end{matrix}$$




$$\rightsquigarrow \begin{pmatrix} 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

one partic soln

# Solving matrix equations

The matrix equation

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has only the trivial solution.

3 pivots.

What does this mean about the matrix equation

$$\begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} f \\ s \\ t \end{pmatrix} = \begin{pmatrix} 20 \\ 1 \\ 1 \end{pmatrix}?$$

<u>one</u> solution.

What does this mean about rabbits?

There is exactly one rabbit population that results in (20,1,1) the following year.

# Section 2.5

## Linear Independence

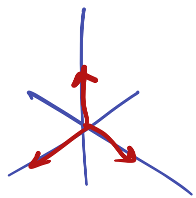
# Section 2.5 Outline

- Understand what is means for a set of vectors to be linearly independent
- Understand how to check if a set of vectors is linearly independent

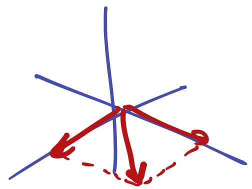
# Linear Independence

The idea of linear independence: a collection of vectors $v_1, \ldots, v_k$ is linearly independent if they are all pointing in truly different directions. Precisely, this means that none of the $v_i$ is in the span of the others.

For example, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are linearly independent.



Also, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$ are linearly dependent.



What is this good for? A basic question we can ask about solving linear equations is: What is the smallest number of vectors needed in the parametric solution to a linear system? We need linear independence to answer this question. See the last slide in this section.

# Linear Independence

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$\underline{x_1}v_1 + \underline{x_2}v_2 + \cdots + \underline{x_k}v_k = 0$$

has only the trivial solution. It is linearly dependent otherwise.

indep $$\underline{0}\begin{pmatrix}1\\0\\0\end{pmatrix} + \underline{0}\begin{pmatrix}0\\1\\0\end{pmatrix} + \underline{0}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

So, linearly dependent means there are $x_1, x_2, \ldots, x_k$ not all zero so that

$$x_1v_1 + x_2v_2 + \cdots + x_kv_k = 0$$

This is a *linear dependence* relation.

$$\underline{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} + \underline{-1}\begin{pmatrix}0\\1\\0\end{pmatrix} + \underline{1}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

dependent.

# Linear Independence

A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

homogen.

has only the trivial solution. no free vars

A 4x3 matrix is 3 vectors in $\mathbb{R}^4$. Those can be indep.

**Fact.** The columns of $A$ are linearly independent
$\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
$\Leftrightarrow$ $A$ has a pivot in each column

Why?

$$A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$$

A 3x4 matrix can't have a pivot in each col so....

4 vectors in $\mathbb{R}^3$ can't be independent

# Linear Independence

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

No! dep.

Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 4 & 10 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 1 & 3 \\ 0 & \boxed{-2} & -2 \\ 0 & 6 & \boxed{6} \end{pmatrix}$$

Indep.

# Linear Independence

When is $\{v\}$ is linearly dependent?

When $V$ = zero vector.

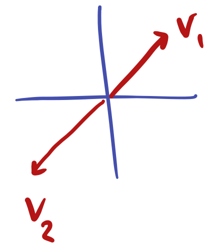
$$\_\_\, V = 0$$

When is $\{v_1, v_2\}$ is linearly dependent?

$$\_\_\, V_1 + \_\_\, V_2 = 0.$$

When $v_1$ is a multiple of $V_2$ (or vice versa)

or: they lie on same line thru $0$.




or: one in span of other.

When is the set $\{v_1, v_2, v_3\}$ linearly dependent?

example

dep $\quad 2 \cdot \begin{pmatrix} 3 \\ 7 \end{pmatrix} - 1 \begin{pmatrix} 6 \\ 14 \end{pmatrix} = 0.$

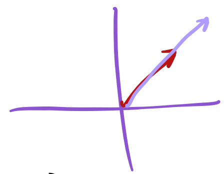
dep $\quad 5 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 6 \\ 14 \end{pmatrix} = 0$

Demo

# Linear Independence

**Fact.** The set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if they span a $k$-dimensional plane. (algebra $\leftrightarrow$ geometry)

**Fact.** The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if we can remove a vector from the set without changing (the dimension of) the span.



**Fact.** The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some $v_i$ lies in the span of $v_1, \ldots, v_{i-1}$.

## Span and Linear Independence

Is $\left\{ \begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Try using the last fact: the set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some $v_i$ lies in the span of $v_1, \ldots, v_{i-1}$.

- $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}$ not in span of previous vectors (there are none)
- $\begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}$ not in span of $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}$ (not a multiple)
- $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ not in span of $\begin{pmatrix} 5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}$ because of the 4

So <u>indep</u>.

# Linear independence and free variables

Theorem. Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$ and consider the vector equation

$$x_1 v_1 + \cdots + x_k v_k = 0.$$

The set of vectors corresponding to non-free variables are linearly independent.

So, given a bunch of vectors $v_1, \ldots, v_k$, if you want to find a collection of $v_i$ that are linearly independent, you put them in the columns of a matrix, row reduce, find the pivots, and then take the original $v_i$ corresponding to those columns.

Example. Try this with $(1, 1, 1)$, $(2, 2, 2)$, and $(1, 2, 3)$.

# Linear independence and coordinates

Fact. If $v_1, \ldots, v_k$ are linearly independent vectors then we can write each element of

$$\mathrm{Span}\{v_1, \ldots, v_k\}$$

in exactly one way as a linear combination of $v_1, \ldots, v_k$.

Fact 1. Say $v_1, \ldots, v_k$ are in $\mathbb{R}^n$. If $k > n$, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

Fact 2. If one of $v_1, \ldots, v_k$ is 0, then $\{v_1, \ldots, v_k\}$ is linearly dependent.

# Parametric vector form and linear independence

**Poll**

Say you find the parametric vector form for a homogeneous system of linear equations, and you find that the set of solutions is the span of certain vectors. Then those vectors are...

1. always linearly independent
2. sometimes linearly independent
3. never linearly independent

Example. In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

# Parametric Vector Forms and Linear Independence

In Section 2.4 we solved the matrix equation $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In parametric vector form, the solution is:

$$x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

The two vectors that appear are linearly independent (why?). This means that we can't write the solution with fewer than two vectors (why?). This also means that this way of writing the solution set is efficient: for each solution, there is only one choice of $x_3$ and $x_4$ that gives that solution.

# Summary of Section 2.5

- A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_k v_k = 0$$

  has only the trivial solution. It is linearly dependent otherwise.
- The cols of $A$ are linearly independent
  $\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
  $\Leftrightarrow$ $A$ has a pivot in each column
- The number of pivots of $A$ equals the dimension of the span of the columns of $A$
- The set $\{v_1, \ldots, v_k\}$ is linearly independent $\Leftrightarrow$ they span a $k$-dimensional plane
- The set $\{v_1, \ldots, v_k\}$ is linearly dependent $\Leftrightarrow$ some $v_i$ lies in the span of $v_1, \ldots, v_{i-1}$.
- To find a collection of linearly independent vectors among the $\{v_1, \ldots, v_k\}$, row reduce and take the (original) $v_i$ corresponding to pivots.

# Typical exam questions

- State the definition of linear independence.
- *Always/sometimes/never.* A collection of 99 vectors in $\mathbb{R}^{100}$ is linearly dependent.
- *Always/sometimes/never.* A collection of 100 vectors in $\mathbb{R}^{99}$ is linearly dependent.
- Find all values of $h$ so that the following vectors are linearly independent:

$$\left\{ \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ h \end{pmatrix} \right\}$$

- *True/false.* If $A$ has a pivot in each column, then the rows of $A$ are linearly independent.
- *True/false.* If $u$ and $v$ are vectors in $\mathbb{R}^5$ then $\{u, v, \sqrt{2}u - \pi v\}$ is linearly independent.
- If you have a set of linearly independent vectors, and their span is a line, how many vectors are in the set?