Announcements Sep 27

- Masks $\rightsquigarrow$ Music?
- Mid-semester survey in Canvas $\to$ Quizzes
- WeBWorK 2.5 & 2.6 due Tuesday nite
- Midterm 2 Oct 20 8–9:15p
- No quiz Friday(?!)
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Find a lin ind span set for Col(A)

$$A = \begin{pmatrix} 1 & \cancel{2} & \cancel{3} & 7 & \cancel{8} \\ 2 & \cancel{4} & \cancel{6} & 8 & \cancel{10} \end{pmatrix}$$

mult of 1st

sum of 1st & 4th

$$\rightsquigarrow \begin{pmatrix} \boxed{1} & 2 & 3 & 7 & 8 \\ 0 & 0 & 0 & \boxed{-6} & -6 \end{pmatrix}$$

x_1 in Nul(A)

x_2 is soln to $Ax=b$

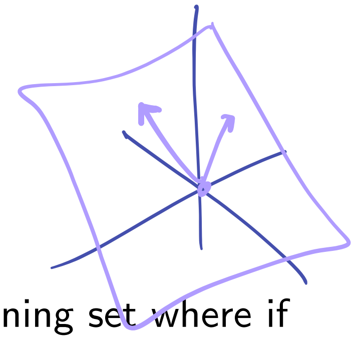
then $A(17x_1+x_2)=b$

Section 2.7

Bases

Bases

V = subspace of $\mathbb{R}^n$ (possibly $V = \mathbb{R}^n$)   plane thru 0,

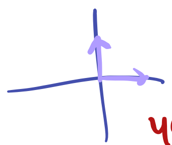


A basis for V is a set of vectors $\{v_1, v_2, \dots, v_k\}$ such that

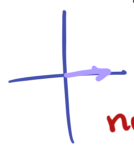
1. $V = \text{Span}\{v_1, \dots, v_k\}$
2. $v_1, \dots, v_k$ are linearly independent

Equivalently, a basis is a *minimal spanning set*, that is, a spanning set where if you remove any one of the vectors you no longer have a spanning set.

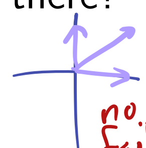
Q. What is one basis for $\mathbb{R}^2$ (V=)? How many bases are there?



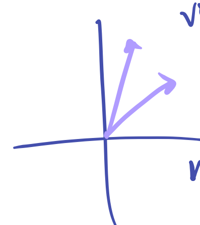
yes



no - fails 1.

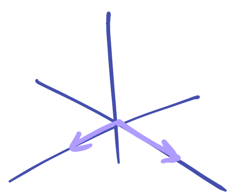


no fails 2



any 2 vectors that are not mult. of each other is a basis

Q. What is one basis for the xy-plane in $\mathbb{R}^3$? Find all bases for the xy-plane.



Any 2 vectors in xy-plane not multiples of each other

Dimension

$V =$ subspace of $\mathbb{R}^n$

$\dim(V) =$ dimension of $V = k =$the number of vectors in the basis

(What is the problem with this definition of dimension?)

How do we know for a fixed V. all bases have same number of vectors?

It turns out: it's ok. All bases have same dim.

Bases for $\mathbb{R}^n$

Let us consider the special case where V is equal to all of $\mathbb{R}^n$.

What are all bases for $V = \mathbb{R}^n$? Or, if we have a set of vectors $\{v_1, \dots, v_k\}$, how do we check if they form a basis for $\mathbb{R}^n$? First, we make them the columns of a matrix....

- For the vectors to be linearly independent we need a pivot in every column.
- For the vectors to span $\mathbb{R}^n$ we need a pivot in every row.

Conclusion: $k = n$ and the matrix has n pivots.

$$\begin{pmatrix} \square & & \\ & \square & \\ & & \square \end{pmatrix}$$

So $\begin{pmatrix} \boxed{5} & 7 & 10 \\ 0 & \boxed{5} & 11 \\ 0 & 0 & \boxed{12} \end{pmatrix}$

cols form a basis for $\mathbb{R}^3$

$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ Too tall

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ Too wide

The standard basis for $\mathbb{R}^n$

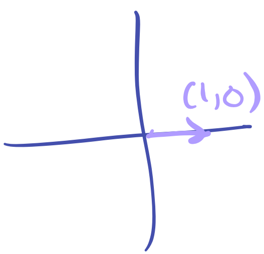
We have the standard basis vectors for $\mathbb{R}^n$:

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \end{aligned}$$

$\underline{\mathbb{R}^2}$

$e_1 = (1, 0)$

$e_2 = (0, 1)$



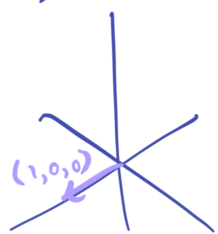
$\underline{\mathbb{R}^3}$

$e_1 = (1, 0, 0)$

$e_2 = (0, 1, 0)$

$e_3 = (0, 0, 1)$

$$\begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$



etc.

Who cares about bases?

A basis $\{v_1, \ldots, v_k\}$ for a subspace V of $\mathbb{R}^n$ is useful because:

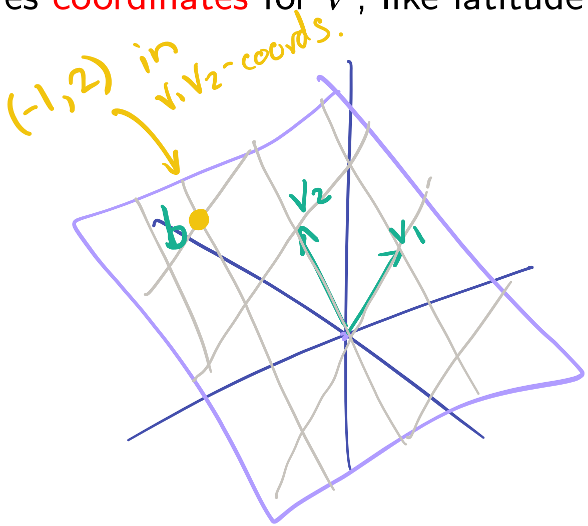
lin ind

Every vector v in V can be written in exactly one way:

$$v = c_1 v_1 + \cdots + c_k v_k$$

span

So a basis gives coordinates for V, like latitude and longitude. See Section 2.8.



Solve $x \cdot v_1 + y \cdot v_2 = b$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{red}]{\text{row}} \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Col(A)

$$\begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Pivot in 1^{st} col $\rightsquigarrow$ So take 1^{st} col of orig. matrix A

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

are pivot cols

Nul(A)

$x = -y - z$

$y = y$

$z = z$

corr to non-pivot cols

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This is the basis.

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Oopsies!

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Nul(A)

$x = z$

$y = -2z$

$z = z$

basis: $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

corr to non-pivot cols

Col(A)

basis $\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$

pivot cols.

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a basis for $\mathrm{Nul}(A)$
- the pivot columns of A form a basis for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

What should you do if you are asked to find a basis for Span$\{v_1, \ldots, v_k\}$?

Find a basis for span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}, \ldots \right.$

Make a matrix where those are the cols. Now it's a Col(A) problem.

Bases for planes

Find a basis for the plane $2x + 3y + z \boxed{= 0}$ in $\mathbb{R}^3$.

makes it a Nul(A) problem

$$\text{Nul}(A) \qquad A = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

$$x = -\tfrac{3}{2}y - \tfrac{1}{2}z$$

$$y = y$$

$$z = z$$

basis for Nul(A)

$$\underbrace{\left\{ \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

so, these span Nul(A) & are linind

Basis theorem

Basis Theorem

If V is a k-dimensional subspace of $\mathbb{R}^n$, then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V, linearly independent, k vectors

So in a 2D plane, a span set with 2 vectors must be lin ind (so we have a basis)

So in a 3D plane a lin ind set of 3 vectors spans the 3D plane (so we have a basis)

We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

Typical exam questions

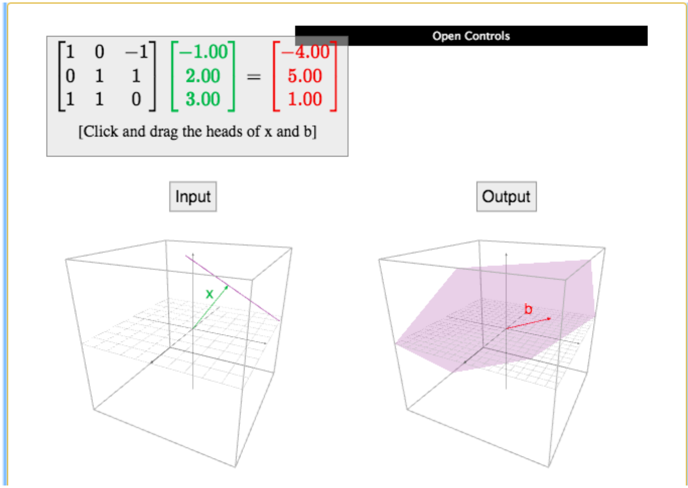
- Find a basis for the yz-plane in $\mathbb{R}^3$
- Find a basis for $\mathbb{R}^3$ where no vector has a zero
- How many vectors are there in a basis for a line in R^7?
- True/false: every basis for a plane in $\mathbb{R}^3$ has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in $\mathbb{R}^3$ and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of A is the number of pivots of A.
- True/false: If b lies in the column space of A, and the columns of A are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span R^3 must be linearly independent.

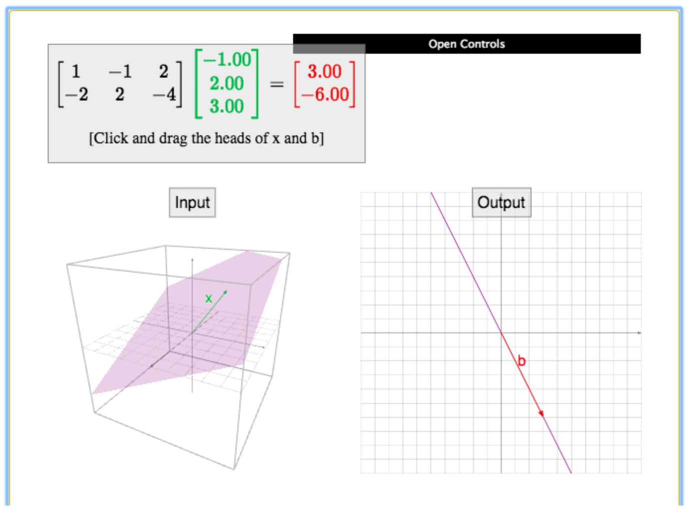
Section 2.9

The rank theorem

Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\mathrm{Col}(A)$:





Rank Theorem

$$\textcolor{red}{\mathrm{rank}(A)} = \dim \mathrm{Col}(A) = \#\text{ pivot columns}$$
$$\textcolor{red}{\mathrm{nullity}(A)} = \dim \mathrm{Nul}(A) = \#\text{ nonpivot columns}$$

Rank Theorem. $\mathrm{rank}(A) + \mathrm{nullity}(A) = \#\mathrm{cols}(A)$

This ties together everything in the whole chapter: rank A describes the b's so that $Ax = b$ is consistent and the nullity describes the solutions to $Ax = 0$. So more flexibility with b means less flexibility with x, and vice versa.

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

About names

Again, why did we need all these vocabulary words? One answer is that the rank theorem would be harder to understand if it was:

The size of a minimal spanning set for the set of solutions to $Ax = 0$ plus the size of a minimal spanning set for the set of b so that $Ax = b$ has a solution is equal to the number of columns of A.

Compare to: $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$

"A common concept in history is that knowing the name of something or someone gives one power over that thing or person." –Loren Graham
http://philoctetes.org/news/the_power_of_names_religion_mathematics

Typical exam questions

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?