Announcements Sep 29

Null space problem

Q. Find a basis for the intersection of the planes

$$x+y+z+w=0$$
$$x+2y+3z+4w=0$$

in $\mathbb{R}^4$

A. Find vector param form

$$A=\begin{pmatrix}1&1&1&1\\1&2&3&4\end{pmatrix}$$

- Masks ⇝ Music!
- Mid-semester survey in Canvas → Quizzes
- WeBWorK 2.7+2.9 & 3.1 due Tuesday nite
- Midterm 2 Oct 20 8–9:15p
- No quiz Friday(?!)
- Use Piazza for general questions
- Office hrs: Tue 4-5 Teams + Thu 1-2 Skiles courtyard/Teams + Pop-ups?
- Many TA office hours listed on Canvas
- Section M web site: Google "Dan Margalit math", click on 1553
  - future blank slides, past lecture slides, advice
- Old exams: Google "Dan Margalit math", click on Teaching
- Tutoring: `http://tutoring.gatech.edu/tutoring`
- PLUS sessions: `http://tutoring.gatech.edu/plus-sessions`
- Math Lab: `http://tutoring.gatech.edu/drop-tutoring-help-desks`
- Counseling center: `https://counseling.gatech.edu`
- You can do it!

Section 2.9

The rank theorem

\# cols with pivots

+ \# cols without pivots

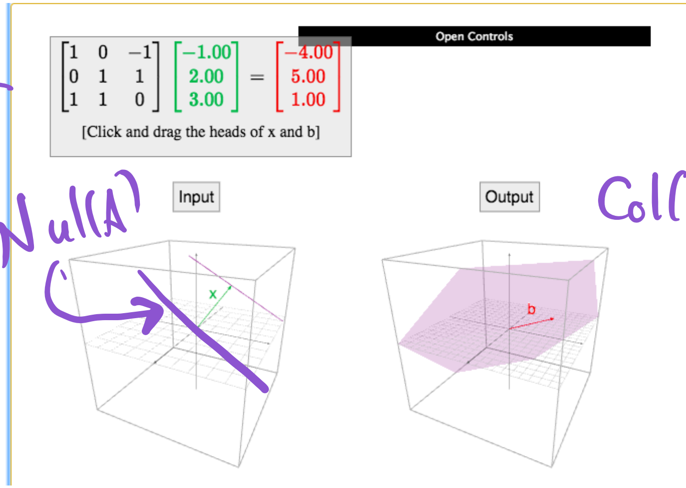
= \# cols

Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\mathrm{Col}(A)$:

2 pivots

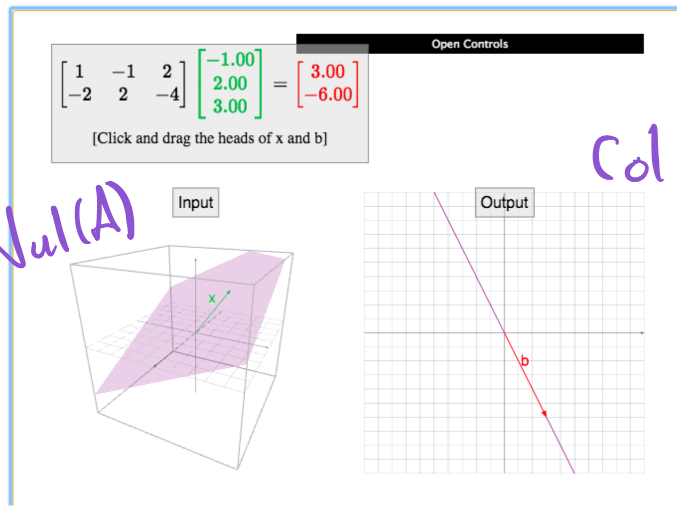
Nul(A)



Col(A)

$1 + 2 = 3$

Nul(A)



Col(A)

$2 + 1 = 3.$

Rank Theorem

#basis vecs.

numbers

$$\text{rank}(A) = \overset{''}{\dim}\,\text{Col}(A) = \#\text{ pivot columns}$$

b/c basis vecs for Col(A) are pivot cols!

$$\text{nullity}(A) = \dim \text{Nul}(A) = \#\text{ nonpivot columns}$$

= # free vars.

Rank Theorem. $\text{rank}(A) + \text{nullity}(A) = \#\text{cols}(A)$

This ties together everything in the whole chapter: rank A describes the b's so that $Ax = b$ is consistent and the nullity describes the solutions to $Ax = 0$. So more flexibility with b means less flexibility with x, and vice versa.

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ \cancel{1} & \cancel{1} & \cancel{1} \\ 1 & \cancel{1} & \cancel{1} \end{pmatrix}$

rank(A) = 1.

Nul(A) = 2

About names

Again, why did we need all these vocabulary words? One answer is that the rank theorem would be harder to understand if it was:

The size of a minimal spanning set for the set of solutions to $Ax = 0$ plus the size of a minimal spanning set for the set of b so that $Ax = b$ has a solution is equal to the number of columns of A.

Compare to: $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$

"A common concept in history is that knowing the name of something or someone gives one power over that thing or person." –Loren Graham
`http://philoctetes.org/news/the_power_of_names_religion_mathematics`

Section 2.9 Summary

- **Rank Theorem.** $\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = \#\mathrm{cols}(A)$

Typical exam questions

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?

Chapter 3

Linear Transformations and Matrix Algebra

Where are we?

In Chapter 1 we learned to solve all linear systems algebraically.

In Chapter 2 we learned to think about the solutions geometrically.

In Chapter 3 we continue with the algebraic abstraction. We learn to think about solving linear systems in terms of inputs and outputs. This is similar to control systems in AE, objects in computer programming, or hot pockets in a microwave.

More specifically, we think of a matrix as giving rise to a function with inputs and outputs. Solving a linear system means finding an input that produces a desired output. We will see that sometimes these functions are invertible, which means that you can reverse the function, inputting the outputs and outputting the inputs.

The invertible matrix theorem is the highlight of the chapter; it tells us when we can reverse the function. As we will see, it ties together everything in the course so far.

Sections 3.1

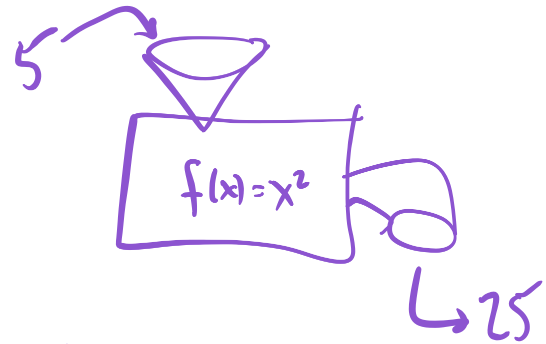
Matrix Transformations

Section 3.1 Outline

- Learn to think of matrices as functions, called matrix transformations
- Learn the associated terminology: domain, codomain, range
- Understand what certain matrices do to $\mathbb{R}^n$

From matrices to functions

A function is a rule with inputs & outputs:



Important: Every input has *one* output

Let A be an $m \times n$ matrix.

We define a function

inputs outputs

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$T(v) = Av$$

vector

This is called a matrix transformation.

another word for function

The domain of T is $\mathbb{R}^n$.

inputs

The co-domain of T is $\mathbb{R}^m$.

potential outputs

The range of T is the set of outputs: $\mathrm{Col}(A)$

actual outputs, always contained in codomain

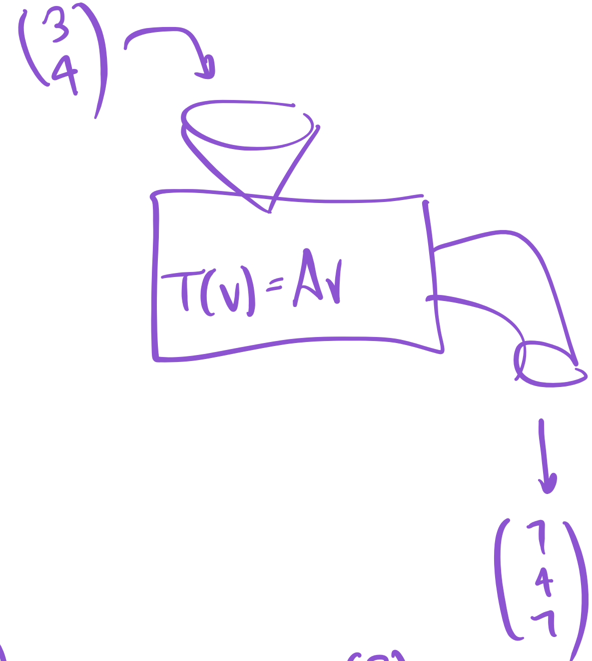
This gives us a*nother* point of view of $Ax = b$

input output

Demo

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$, $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$.



What is $T(u)$? $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 7 \end{pmatrix}$

Find v in $\mathbb{R}^2$ so that $T(v) = b$

input

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$ answer: $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

or

$\left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right)$

Find a vector in $\mathbb{R}^3$ that is not in the range of T.

Any vector not in Col(A)

Any vector where top & bottom numbers not same. eg. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Square matrices

For a square matrix $n \times n$ we can think of the associated matrix transformation

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

as doing something to $\mathbb{R}^n$.

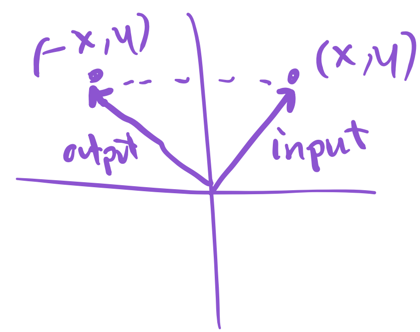
input

Example. The matrix transformation T for

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

What does T do to $\mathbb{R}^2$?

~~Reflection~~ Flip over y-axis

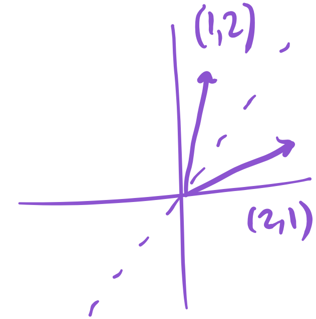


Square matrices

What does each matrix do to $\mathbb{R}^2$?

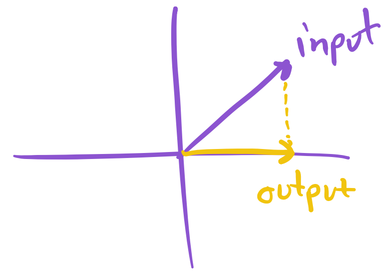
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

flip/reflect over $y=x$



$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

(orthogonal) projection to x-axis



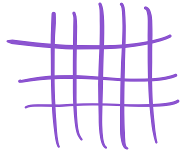
$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

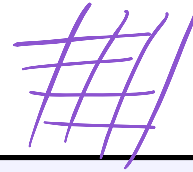
dilating by 3

What is the range in each case?

$\mathbb{R}^2$, x-axis, $\mathbb{R}^2$

Poll



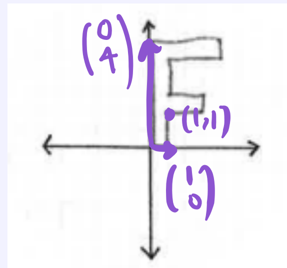


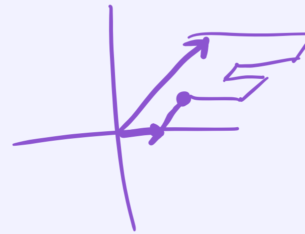
Poll

What does $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ do to this letter F?

italicize!

or shear





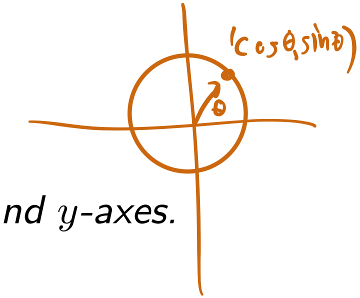
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 6 \end{pmatrix} = \begin{pmatrix} x \\ 6 \end{pmatrix}$$

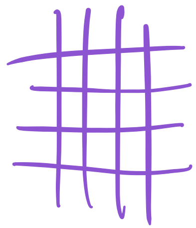
Square matrices

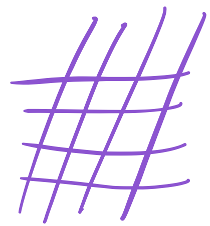


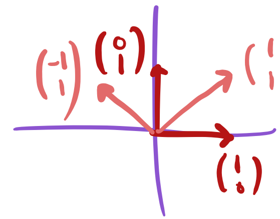
What does each matrix do to $\mathbb{R}^2$?

Hint: if you can't see it all at once, see what happens to the x- and y-axes.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ shear







$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

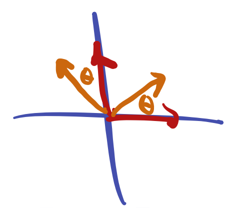
rotate by $\pi/4$ & scale by $\sqrt{2}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

rotation by θ



Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

(orthogonal) projection to xy-plane.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Why are we learning about matrix transformations?

Sample applications:

- Cryptography (Hill cypher)
- Computer graphics (Perspective projection is a linear map!)
- Aerospace (Control systems - input/output)
- Biology
- Many more!

shear

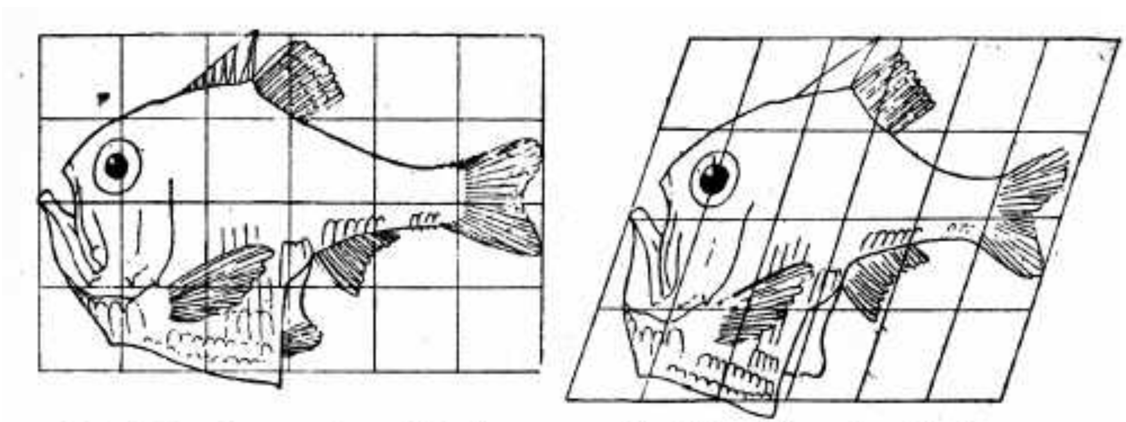


Fig. 517. *Argyropelecus Olfersi.* Fig. 518. *Sternoptyx diaphana.*

Applications of Linear Algebra

Biology: In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population in 2016 (in terms of the number of first, second, and third year rabbits), then what is the population in 2017?

These relations can be represented using a matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

How does this relate to matrix transformations?

Demo

Section 3.1 Summary

- If A is an $m \times n$ matrix, then the associated matrix transformation T is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If A is $n \times n$ then T does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

Typical exam questions

- What does the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ do to $\mathbb{R}^2$?
- What does the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ do to $\mathbb{R}^3$?
- What does the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ do to $\mathbb{R}^2$?
- True/false. If A is a matrix and T is the associated matrix transformation, then the statement $Ax = b$ is consistent is equivalent to the statement that b is in the range of T.
- True/false. There is a matrix A so that the domain of the associated matrix transformation is a line in $\mathbb{R}^3$.