1.3 Parametric Form

Outline of Section 1.3

- Find the parametric form for the solutions to a system of linear equations.
- Describe the geometric picture of the set of solutions.

Free Variables

We know how to understand the solution to a system of linear equations when every column to the left of the vertical line has a pivot. For instance:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

If the variables are x and y what are the solutions?

Free Variables

How do we solve a system of linear equations if the row reduced matrix has a column without a pivot? For instance:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

represents two equations:

$$\begin{aligned} x_1 \qquad + 5x_3 &= 0 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

There is one free variable x_3, corresponding to the non-pivot column. To solve, we move the free variable to the right:

$$\begin{aligned} x_1 \qquad\qquad &= -5x_3 \\ x_2 \qquad &= 1 - 2x_3 \\ x_3 &= x_3 \text{ (free; any real number)} \end{aligned}$$

This is the parametric solution. We can also write the solution as:

$$(-5x_3, 1 - 2x_3, x_3)$$

What is one particular solution? What does the set of solutions look like?

Free Variables

Solve the system of linear equations in x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 \quad + 5x_3 \quad &= 0 \\ x_4 &= 0 \end{aligned}$$

So the associated matrix is:

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

To solve, we move the free variable to the right:

$$\begin{aligned} x_1 &= -5x_3 \\ x_2 &= \quad x_2 \quad \text{(free)} \\ x_3 &= \quad x_3 \quad \text{(free)} \\ x_4 &= 0 \end{aligned}$$

Or: $(-5x_3, x_2, x_3, 0)$. This is a plane in $\mathbb{R}^4$.

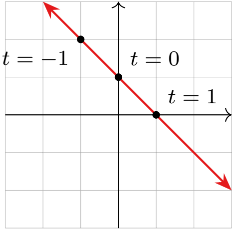
The original equations are the implicit equations for the solution. The answer to this question is the parametric solution.

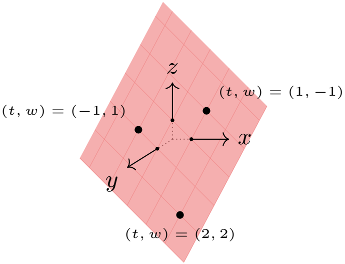
Free variables

Geometry

If we have a consistent system of linear equations, with n variables and k free variables, then the set of solutions is a k-dimensional plane in $\mathbb{R}^n$.

Why does this make sense?





Poll

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. two points
4. line
5. plane
6. 3-dimensional plane
7. 4-dimensional plane

Implicit versus parametric equations of planes

Find a parametric description of the plane

$$x + y + z = 1$$

The original version is the implicit equation for the plane. The answer to this problem is the parametric description.

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of system of linear equations.

1. The last column is a pivot column.
   $\rightsquigarrow$ the system is *inconsistent*.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   $\rightsquigarrow$ the system has a *unique solution*.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   $\rightsquigarrow$ the system has *infinitely many* solutions; free variables correspond to columns without pivots.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Typical exam questions

True/False: If a system of equations has 100 variables and 70 equations, then there must be infinitely many solutions.

True/False: If a system of equations has 70 variables and 100 equations, then it must be inconsistent.

How can we tell if an augmented matrix corresponds to a consistent system of linear equations?

If a system of linear equations has finitely many solutions, what are the possible numbers of solutions?