

Section 2.7

Bases

Outline of Section 2.7

- Definition of basis
- Definition of dimension
- The standard basis for \mathbb{R}^n
- Bases for $\text{Col}(A)$ and $\text{Nul}(A)$
- Basis theorem: from basis to linear independence or spanning

Bases

$V =$ subspace of \mathbb{R}^n (possibly $V = \mathbb{R}^n$)

A **basis** for V is a set of vectors $\{v_1, v_2, \dots, v_k\}$ such that

1. $V = \text{Span}\{v_1, \dots, v_k\}$
2. v_1, \dots, v_k are linearly independent

Equivalently, a basis is a *minimal spanning set*, that is, a spanning set where if you remove any one of the vectors you no longer have a spanning set.

Q. What is one basis for \mathbb{R}^2 ? How many bases are there?

Q. What is one basis for the xy -plane in \mathbb{R}^3 ? Find all bases for the xy -plane.

Dimension

$V =$ subspace of \mathbb{R}^n

$\dim(V) =$ **dimension** of $V = k =$ the number of vectors in the basis

(What is the problem with this definition of dimension?)

Bases for \mathbb{R}^n

Let us consider the special case where V is equal to all of \mathbb{R}^n .

What are all bases for $V = \mathbb{R}^n$? Or, if we have a set of vectors $\{v_1, \dots, v_k\}$, how do we check if they form a basis for \mathbb{R}^n ? First, we make them the columns of a matrix....

- For the vectors to be linearly independent we need a **pivot in every column**.
- For the vectors to span \mathbb{R}^n we need a **pivot in every row**.

Conclusion: $k = n$ and the matrix has n pivots.

The standard basis for \mathbb{R}^n

We have the standard basis vectors for \mathbb{R}^n :

$$e_1 = (1, 0, 0, \dots, 0)$$

$$e_2 = (0, 1, 0, \dots, 0)$$

\vdots

Who cares about bases?

A basis $\{v_1, \dots, v_k\}$ for a subspace V of \mathbb{R}^n is useful because:

Every vector v in V can be written in exactly one way:

$$v = c_1v_1 + \cdots + c_kv_k$$

So a basis gives **coordinates** for V , like latitude and longitude. See Section 2.8.

Bases for $\text{Nul}(A)$ and $\text{Col}(A)$

Find bases for $\text{Nul}(A)$ and $\text{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



Bases for $\text{Nul}(A)$ and $\text{Col}(A)$

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Bases for $\text{Nul}(A)$ and $\text{Col}(A)$

Find bases for $\text{Nul}(A)$ and $\text{Col}(A)$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Bases for $\text{Nul}(A)$ and $\text{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a basis for $\text{Nul}(A)$
- the pivot columns of A form a basis for $\text{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

What should you do if you are asked to find a basis for $\text{Span}\{v_1, \dots, v_k\}$?

Bases for planes

Find a basis for the plane $2x + 3y + z = 0$ in \mathbb{R}^3 .

Basis theorem

Basis Theorem

If V is a k -dimensional subspace of \mathbb{R}^n , then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V , linearly independent, k vectors

We are skipping Section 2.8 this semester. But remember: the whole point of a basis is that it gives coordinates (like latitude and longitude) for a subspace. Every point has a unique address.

Section 2.7 Summary

- A **basis** for a subspace V is a set of vectors $\{v_1, v_2, \dots, v_k\}$ such that
 1. $V = \text{Span}\{v_1, \dots, v_k\}$
 2. v_1, \dots, v_k are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\text{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\text{Col}(A)$ by taking pivot columns of A (not reduced A)
- **Basis Theorem.** Suppose V is a k -dimensional subspace of \mathbb{R}^n . Then
 - ▶ Any k linearly independent vectors in V form a basis for V .
 - ▶ Any k vectors in V that span V form a basis.

Typical exam questions

- Find a basis for the yz -plane in \mathbb{R}^3
- Find a basis for \mathbb{R}^3 where no vector has a zero
- How many vectors are there in a basis for a line in \mathbb{R}^7 ?
- True/false: every basis for a plane in \mathbb{R}^3 has exactly two vectors.
- True/false: if two vectors lie in a plane through the origin in \mathbb{R}^3 and they are not collinear then they form a basis for the plane.
- True/false: The dimension of the null space of A is the number of pivots of A .
- True/false: If b lies in the column space of A , and the columns of A are linearly independent, then $Ax = b$ has infinitely many solutions.
- True/false: Any three vectors that span \mathbb{R}^3 must be linearly independent.