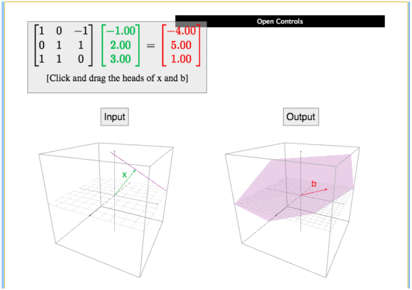
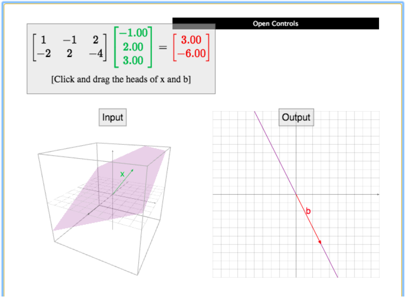
Section 2.9

The rank theorem

Rank Theorem

On the left are solutions to $Ax = 0$, on the right is $\mathrm{Col}(A)$:





Rank Theorem

$$\text{rank}(A) = \dim \text{Col}(A) = \#\text{ pivot columns}$$
$$\text{nullity}(A) = \dim \text{Nul}(A) = \#\text{ nonpivot columns}$$

Rank Theorem. $\text{rank}(A) + \text{nullity}(A) = \#\text{cols}(A)$

This ties together everything in the whole chapter: rank A describes the b's so that $Ax = b$ is consistent and the nullity describes the solutions to $Ax = 0$. So more flexibility with b means less flexibility with x, and vice versa.

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

About names

Again, why did we need all these vocabulary words? One answer is that the rank theorem would be harder to understand if it was:

The size of a minimal spanning set for the set of solutions to $Ax = 0$ plus the size of a minimal spanning set for the set of b so that $Ax = b$ has a solution is equal to the number of columns of A.

Compare to: $\text{rank}(A) + \text{nullity}(A) = n$

“A common concept in history is that knowing the name of something or someone gives one power over that thing or person.” –Loren Graham
http://philoctetes.org/news/the_power_of_names_religion_mathematics

Section 2.9 Summary

- Rank Theorem. $\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = \#\operatorname{cols}(A)$

Typical exam questions

- Suppose that A is a 5×7 matrix, and that the column space of A is a line in $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the column space of A is $\mathbb{R}^5$. Describe the set of solutions to $Ax = 0$.
- Suppose that A is a 5×7 matrix, and that the null space is a plane. Is $Ax = b$ consistent, where $b = (1, 2, 3, 4, 5)$?
- True/false. There is a 3×2 matrix so that the column space and the null space are both lines.
- True/false. There is a 2×3 matrix so that the column space and the null space are both lines.
- True/false. Suppose that A is a 6×2 matrix and that the column space of A is 2-dimensional. Is it possible for $(1, 0)$ and $(1, 1)$ to be solutions to $Ax = b$ for some b in $\mathbb{R}^6$?